Stability of geodesics on three-dimensional Lie groups

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Abstract

In Riemannian geometry, a ***geodesic*** is the straightest curve as well as the shortest curve between any two close points. Geodesics are analogues of straight lines in Euclidean geometry, and as such, are probably the most "important" curves on a Riemannian manifold. Despite that, geodesic computation involves solving a system of non-linear second-order ODEs, which is a difficult task in general. The difficulty is significantly reduced in the case of a metric Lie group G, where the geodesic can be found by the *Euler-Arnold* equation [Arnold, 1966],[Euler, 1765]. This approach involves solving a system of first-order ODEs that is a little more tractable. In this project, we will study the behavior of these geodesics in the first non-trivial dimension, $n = 3$. We will start by classifying unimodular three-dimensional Lie group using Milnor's method [Milnor, 1976], and continue to explicitly solve the Euler-Arnold equation in each case. Furthermore, we completely classify stable and unstable equilibria of the equation in the sense of stability analysis for ODEs.

1 Introduction

In his famous [Arnold, 1966], Arnold showed that the laws that governed the motion of a rigid body and an incompressible, inviscid fluid are very much similar: they both follow the geodesic flow of a one-sided invariant metric on a Lie group. This geodesic motion is formulated by the Euler-Arnold equation. Since then, there have been various attempts to generalise the equation and apply it to survey PDEs such as Burgers' equation, KDV equation, and Camassa-Holm equation (see, for example, [Vizman, 2008]). In this report, we use the Euler-Arnold equation of geodesics for a particular class of metric Lie groups. A motivation comes from the important role of geodesics in Riemannian geometry, in which a geodesic is a natural generalisation of a straight line in Euclidean geometry. In general Riemannian settings, solving for geodesics is a difficult task as it involves solving a system of second-order ODEs that are non-linear in the first derivatives. However, for a homogeneous manifold such as a Lie group G with a left-invariant metric, geodesics can be found from the Euler-Arnold equation via a system of first-order ODEs which is easier to handle (see 4.1 below). The paper is organised as follows. Section 2 introduces necessary concepts and examples in the theory of Lie algebras; Section 3 gives the classification of 3-dimensional unimodular Lie groups; and in Section 4 we derive the geodesics for each case of the classification, as well as investigate their stability. Our main result, the complete classification of stable and non-stable equilibria of the Euler-Arnold equation for 3-dimensional unimodular metric Lie groups is given in Section 5, which also contains some open questions and a discussion of directions for further research.

2 Lie algebras: Definitions and Examples

First of all, we introduce Lie groups via *Lie algebra* and *Lie brackets* [Erdmann and Wildon, 2006].

Definition 2.1. Let $\mathbb{F}$ be a field. A ***Lie algebra*** over $\mathbb{F}$ is an $\mathbb{F}$-vector space L, together with a bilinear map, the ***Lie bracket***

$$L \times L \to L, \qquad (x, y) \to [x, y]$$

having the following properties:

$$[x, x] = 0, \quad \text{for all } x \in L,$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \text{for all } x, y, z \in L.$$

The second condition is known as the *Jacobi identity.* As the Lie bracket $[-, -]$ is bilinear, we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Hence the first condition implies anticommutativity:

$$[x, y] = -[y, x], \quad \text{for all } x, y \in L.$$

Using the defined Lie brackets, we can easily classify Lie algebras of dimensions less than 3.

Example 2.1. Suppose that L is a 1-dimensional Lie algebra. Then any vector of L is of the form cX, for some non-zero $x \in L$. By bilinearity and alternativity, we have

$$[x, cx] = c \cdot [x, x] = c \cdot 0 = 0.$$

A Lie algebra is said to be ***abelian*** if the Lie bracket of any two elements in it is zero. So it is clear that any 1-dimensional Lie algebra is abelian.

Example 2.2. Suppose that L is a 2-dimensional Lie algebra. We start with a basis $\{x, y\}$ of L, then $[x, y] = ax + by$, for some $a, b \in \mathbb{F}$.

(1) If $a = 0$ and $b = 0$, then $[x, y] = 0$ and L is abelian.

(2) If $[x, y] \neq 0$, then either $a \neq 0$ or $b \neq 0$. Up to renaming the basis vectors we can assume that $a \neq 0$. Let $z = [x, y]$ and consider

$$[z, y] = [ax + by, y] = a[x, y] + b[y, y] = a[x, y] = az.$$

Take $w = a^{-1}y$, then $[z, w] = aa^{-1}z = z$. So, by contruction, it is shown that any 2-dimensional non-abelian Lie algebra has a basis $\{x, y\}$ such that its Lie bracket satisfies $[x, y] = x$.

It is important to give some further definitions and examples of Lie algebras that will be used later in the report.

Example 2.3. A well-known example of a Lie algebra is $\mathbb{R}^3$ with the Lie bracket defined by the cross product. This is a particular case of the general construction which we further study in Section 3.

Example 2.4. For finite-dimensional vector space V over $\mathbb{F}$, the ***general linear algebra*** $\mathfrak{gl}(V)$ of V is the Lie algebra whose elements are linear maps from V to V, and whose Lie bracket is given by

$$[x, y] = x \circ y - y \circ x \text{ for } x, y \in \mathfrak{gl}(V)$$

where $\circ$ denotes the composition of maps.

Example 2.5. For all $n \times n$ matrices over $\mathbb{F}$, $\mathfrak{gl}(n)$ is a Lie algebra whose Lie bracket is defined by

$$[A, B] = AB - BA$$

where AB is the usual product of the matrices A and B.
If $\dim V = n$, the algebras $\mathfrak{gl}(n)$ and $\mathfrak{gl}(V)$ over the same field are isomorphic.

Definition 2.2. If L_1 and L_2 are Lie algebras over a field F, then a linear map $\rho : L_1 \to L_2$ is called a ***homomorphism*** if

$$\rho([x, y]) = [\rho(x), \rho(y)] \text{ for all } x, y \in L_1.$$

It is crucial to mention an extremely important homomorphism before we move on.

Definition 2.3. For a Lie algebra L and any $x \in L$ we define a map

$$\mathrm{ad}_x : L \to L, \qquad y \mapsto [x, y]$$

which is called the ***adjoint action***.
The ***adjoint homomorphism*** $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is defined by

$$x \mapsto \mathrm{ad}_x, \qquad \text{for } x \in L.$$

Definition 2.4. A Lie algebra L is said to be ***unimodular*** if $\mathrm{tr}(\mathrm{ad}_x) = 0$ for all $x \in L$.

Example 2.6.

1. It can be seen that for an abelian Lie algebra, $\mathrm{ad}_x = 0$ for all x. Hence $\mathrm{tr}(\mathrm{ad}_x) = 0$ for all x, hence any abelian Lie algebra is unimodular.

2. For the 2-dimensional non-abelian Lie algebra, choose a basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_1$ (see Example 2.2.)
   Then $\mathrm{ad}_{e_1}(e_1) = [e_1, e_1] = 0$, and $\mathrm{ad}_{e_1}(e_2) = [e_1, e_2] = e_1$. And so we get $\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, relative to our basis.
   Similarly, we have $\mathrm{ad}_{e_2}(e_1) = [e_2, e_1] = -e_1$, and $\mathrm{ad}_{e_2}(e_2) = [e_2, e_2] = 0$. Hence we obtain $\mathrm{ad}_{e_2} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, relative to our basis.
   So we conclude that the 2-dimensional non-abelian Lie algebra is not unimodular.

3. One can show that the algebra $\mathfrak{gl}(V)$ is unimodular, as also is the algebra $\mathbb{R}^3$ with the bracket defined by the cross product.

3 Classification of unimodular 3-dimensional Lie algebras

Now we can start the classification of 3-dimensional metric Lie algebras over $\mathbb{R}$. We begin by a lemma from [Milnor, 1976].

Lemma 3.1. *Let L be a 3-dimensional Lie algebra over $\mathbb{R}^3$, where in $\mathbb{R}^3$ we have a fixed inner product and an orientation. Then there exists a unique linear map A from L to itself such that the Lie bracket is related to the cross product by*

$$[u, v] = A(u \times v).$$

The Lie algebra L is unimodular if and only if the map A is self-adjoint.

Proof. Choose an oriented orthonormal basis e_1, e_2, e_3 and define the linear map $A : L \to L$ by $A(e_1) = [e_2, e_3], A(e_2) = [e_3, e_1], A(e_3) = [e_1, e_2]$. Then $A(e_i \times e_j) = [e_i, e_j]$, hence $A(u \times v) = [u, v]$ by bilinearity of both the Lie bracket and the cross product.

Introduce $\alpha_{ij} \in \mathbb{R}$, where $i, j \in \{1, 2, 3\}$, by

$$A(e_i) = \sum \alpha_{ij} e_j$$

which gives:

$$\begin{aligned} A(e_1) &= \alpha_{11}e_1 + \alpha_{12}e_2 + \alpha_{13}e_3 \\ A(e_2) &= \alpha_{21}e_1 + \alpha_{22}e_2 + \alpha_{23}e_3 \\ A(e_3) &= \alpha_{31}e_1 + \alpha_{32}e_2 + \alpha_{33}e_3 \end{aligned}$$

Then we have:

$$\begin{aligned} \mathrm{ad}_{e_1}(e_1) &= [e_1, e_1] = 0 \\ \mathrm{ad}_{e_1}(e_2) &= [e_1, e_2] = A(e_3) = \alpha_{31}e_1 + \alpha_{32}e_2 + \alpha_{33}e_3 \\ \mathrm{ad}_{e_1}(e_3) &= [e_1, e_3] = -[e_3, e_1] = -A(e_2) = -\alpha_{21}e_1 - \alpha_{22}e_2 - \alpha_{23}e_3 \end{aligned}$$

So we obtain $\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & \alpha_{31} & -\alpha_{21} \\ 0 & \alpha_{32} & -\alpha_{22} \\ 0 & \alpha_{33} & -\alpha_{23} \end{pmatrix}$, and so $\mathrm{tr}(\mathrm{ad}_{e_1}) = \alpha_{32} - \alpha_{23}$.

Likewise, we get $\mathrm{tr}(\mathrm{ad}_{e_2}) = \alpha_{13} - \alpha_{31}$ and $\mathrm{tr}(\mathrm{ad}_{e_3}) = \alpha_{21} - \alpha_{12}$.

Thus, L is unimodular if and only if the matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$ of A relative to our basis is symmetric: $\alpha_{ij} = \alpha_{ji}$ for every i, j, that is, A is a self-adjoint linear map. □

We will focus on the unimodular condition established by Lemma 3.1. If A is self-adjoint, then there exists an orthonormal basis e_1, e_2, e_3 consisting of the eigenvectors of A: $A(e_i) = \lambda_i e_i$. Then we have

$$[e_1, e_2] = A(e_3) = \lambda_3 e_3,$$
$$[e_2, e_3] = A(e_1) = \lambda_1 e_1,$$
$$[e_3, e_1] = A(e_2) = \lambda_2 e_2.$$

Remark 3.1. The three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are apparently defined up to orientation. This means that if we change the orientation of our basis, then A changes to $-A$, and so we can switch sign of all three eigenvalues without changing the properties of the Lie algebra. For example, we can choose $\tilde{e_1} = -e_1, \tilde{e_2} = -e_2, \tilde{e_3} = -e_3$, and that gives

$$[\tilde{e_1}, \tilde{e_2}] = [-e_1, -e_2] = [e_1, e_2] = \lambda_3 e_3 = -\lambda_3 \tilde{e_3},$$
$$[\tilde{e_2}, \tilde{e_3}] = [-e_2, -e_3] = [e_2, e_3] = \lambda_1 e_1 = -\lambda_1 \tilde{e_1},$$
$$[\tilde{e_3}, \tilde{e_1}] = [-e_3, -e_1] = [e_3, e_1] = \lambda_2 e_2 = -\lambda_2 \tilde{e_2}$$

Hence all unimodular 3-dimensional metric Lie algebras can be determined by choosing the sign, $+, -$ or 0 for every element of the triple $(\lambda_1, \lambda_2, \lambda_3)$, up to cyclic permutation and simultaneous change of the sign to the opposite.

In conclusion, we have 6 classes of metric unimodular 3-dimensional Lie algebras as follows in the table.

$(\lambda_1, \lambda_2, \lambda_3)$	Similar variations
$(0, 0, 0)$	
$(+, 0, 0)$	$(0, +, 0), (0, 0, +), (-, 0, 0), (0, -, 0), (0, 0, -)$
$(+, +, 0)$	$(0, +, +), (+, 0, +), (-, -, 0), (0, -, -), (-, 0, -)$
$(+, -, 0)$	$(0, +, -), (-, 0, +), (-, +, 0), (0, -, +), (+, 0, -)$
$(+, +, +)$	$(-, -, -)$
$(+, +, -)$	$(-, +, +), (+, -, +), (-, -, +), (+, -, -), (-, +, -)$

Remark 3.2. In the study of equilibria and stability which is the main aim of our project, we will mostly be interested in the signs of $\lambda_1, \lambda_2, \lambda_3$ rather than in the absolute values. As explained in [Milnor, 1976], the algebras with the same "sign structure" are isomorphic as Lie algebras, but may not be isomorphic as metric Lie algebras. To see that, we can scale each of the vectors e_1, e_2, e_3 in such a way that all positive λ_i become $+1$, all negative λ_i become -1 (and of course all zeros remain zeros). So the Lie brackets for two algebras will be the same provided the "sign structure" are the same, but the resulting bases e_1, e_2, e_3 will no longer be orthonormal, only orthogonal.

4 Geodesics and stability analysis on unimodular 3-dimensional Lie groups

For a curve $\gamma(t)$ in a Lie group G, let $X = X(t)$ be defined as the image of the vector $\dot{\gamma}(t)$ under the tangent map of the left multiplication by $(\gamma(t))^{-1}$ in G. Note that $X(t)$ is a curve in the Lie algebra L of G. We introduce the important Euler-Arnold equation [Arnold, 1966],[Euler, 1765].

Theorem 4.1 (Euler-Arnold equation)**.** *The curve $\gamma(t)$ is a naturally parametrised geodesic on a Lie group G if and only if*

$$\dot{X} = \mathrm{ad}^t_X X \tag{1}$$

where ad^t_X is the metric adjoint to ad_X.

Remark 4.1. The system (1) has a first integral $\frac{1}{2}\|X\|^2$, which in the three-dimensional case means

$$X_1^2 + X_2^2 + X_3^2 = constant.$$

This is easy to prove since we have

$$\left(\frac{1}{2}\|X\|^2\right)^{\cdot} = \left(\frac{1}{2}\langle X, X\rangle\right)^{\cdot} = \langle \dot{X}, X\rangle = \langle \mathrm{ad}^t_X X, X\rangle = \langle X, \mathrm{ad}_X X\rangle = 0.$$

Definition 4.1. An ***equilibrium solution*** X_0 of system 1 is a constant solution. It can be found by equating all three expressions on the right-hand sides of 1 to zero.

Definition 4.2. An equilibrium solution X_0 is called ***stable*** if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any solution $X(t), t > 0$, we have

$$\|X(0) - X_0\| < \delta \Rightarrow \|X(t) - X_0\| < \epsilon.$$

Informally, every solution of the ODE that starts close to the stable equilibrium at $t = 0$ would remain close to the equilibrium for all future time.

Using 6 cases of 3-dimensional unimodular metric Lie algebras, we can investigate the existence of equilibrium geodesics in each of them, as well as their stability.

Case 1: $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$
This is an abelian Lie algebra, so we have $[X, Y] = 0$ for any X, Y.
Hence

$$\dot{X}(t) = \mathrm{ad}^t_X X = 0$$
$$\Rightarrow X(t) = X_0$$

So all points are equilibria and are **stable**.

Case 2: $(\lambda_1, \lambda_2, \lambda_3) = (+, 0, 0)$

We have

$$\begin{aligned} X(t) &= X_1 e_1 + X_2 e_2 + X_3 e_3 \\ \mathrm{ad}_{X(t)} &= X_1 \mathrm{ad}_{e_1} + X_2 \mathrm{ad}_{e_2} + X_3 \mathrm{ad}_{e_3} \\ &= \begin{pmatrix} 0 & -\lambda_1 X_3 & \lambda_1 X_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The equation 1 gives

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda_1 X_1 X_3 \\ \lambda_1 X_1 X_2 \end{pmatrix}$$

or

$$\begin{cases} \dot{X}_1 = 0 \\ \dot{X}_2 = -\lambda_1 X_1 X_3 \\ \dot{X}_3 = \lambda_1 X_1 X_2 \end{cases} \Rightarrow \begin{cases} X_1 = c \\ X_2 = -c_1 \sin(\lambda_1 ct) + c_2 \cos(\lambda_1 ct) \\ X_3 = c_1 \cos(\lambda_1 ct) + c_2 \sin(\lambda_1 ct) \end{cases}$$

Equilibrium solutions are given by

$$\begin{cases} -\lambda_1 X_1 X_3 = 0 \\ \lambda_1 X_1 X_2 = 0 \end{cases} \Rightarrow X_1 = 0 \quad \text{or} \quad X_2 = X_3 = 0.$$

(1) From 4.1, the curve $X(t)$ is defined on a sphere, and $X_2 = X_3 = 0$ gives the north pole and the south pole of the sphere. Consider the unit sphere for simplicity, the north pole is $X_0 = (1, 0, 0)$ and $X(0) = (c, c_2, c_1)$. So

$$\|X(0) - X_0\| = \sqrt{(1-c)^2 + c_1^2 + c_2^2}$$

Also, we have

$$\|X(t) - X_0\| = \sqrt{(1-c)^2 + c_1^2 + c_2^2}$$

So if we choose $\delta = \epsilon$, then the stability condition is satisfied. So these equilibria are **stable**. This agrees with the observation that every solution to 1 travels on parallels of the sphere, so the distance to the poles remains constant all the time.

(2) The equilibria $X_1 = 0$ gives the equator of the sphere. Fix a point on the equator, then every solution travelling on parallels of the sphere will not be close to this fixed point, so these equilibria are **unstable.**

Case 3: $(\lambda_1, \lambda_2, \lambda_3) = (+, +, 0)$

Computing in a similar manner, we obtain a system of ODEs:

$$\begin{cases} \dot{X}_1 = \lambda_2 X_2 X_3 \\ \dot{X}_2 = -\lambda_1 X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases} \quad \text{, for } \lambda_1, \lambda_2 > 0.$$

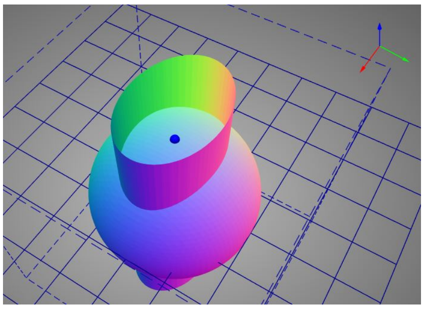
Equilibrium solutions are given by

$$\begin{cases} \lambda_2 X_2 X_3 = 0 \\ -\lambda_1 X_1 X_3 = 0 \\ (\lambda_1 - \lambda_2) X_1 X_2 = 0 \end{cases} \Rightarrow X_1 = X_2 = 0 \quad \text{or} \quad X_2 = X_3 = 0 \quad \text{or} \quad X_3 = X_1 = 0.$$

Cross-multiplying the first two ODEs yields:

$$-\lambda_1 X_3 X_1 \dot{X}_1 = \lambda_2 X_3 X_2 \dot{X}_2$$
$$\Rightarrow \lambda_1 X_1 \dot{X}_1 + \lambda_2 X_2 \dot{X}_2 = 0$$
$$\Rightarrow \lambda_1 X_1^2 + \lambda_2 X_2^2 = c_1$$

This implies that the solution of this system can be viewed as the intersection between a sphere and an elliptic cylinder.

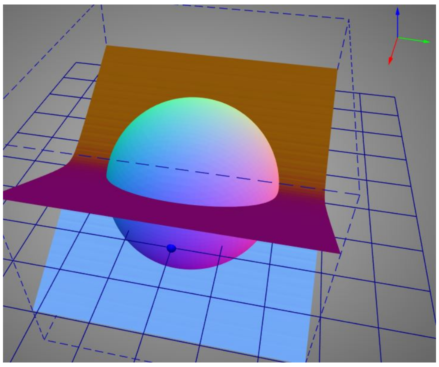


We see that the plotted equilibrium $X_1 = X_2 = 0$ is **stable**, as the solution will keep at a close distance to this point.

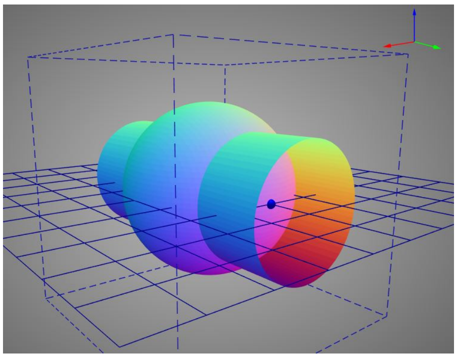
Similarly, the other two pairs of ODEs gives:

$$(\lambda_1 - \lambda_2) X_1^2 - \lambda_2 X_3^2 = c_2, \quad \text{and}$$
$$(\lambda_1 - \lambda_2) X_2^2 + \lambda_1 X_3^2 = c_3$$

- If $\lambda_1 > \lambda_2$, then $\lambda_1 - \lambda_2 > 0$. The first relation defines a hyperbolic cylinder, and the intersection is shown in the figure below.



The plotted equilibrium $X_2 = X_3 = 0$ is **unstable**, as the distance from a point on the intersection curve to this equilibrium point will not remain sufficiently small. On the other hand, the second relation defines an elliptic cylinder (this case is shown below)



So similar to the above analysis, this equilibrium $X_3 = X_1 = 0$ is **stable**.

- If $\lambda_1 < \lambda_2$, then $\lambda_1 - \lambda_2 < 0$. The first relation defines now an elliptic cylinder, and the equilibrium $X_2 = X_3 = 0$ is **stable**. The second relation defines now a hyperbolic cylinder, and the equilibrium $X_3 = X_1 = 0$ is **unstable**.

- If $\lambda_1 = \lambda_2$, then the system degenerates back to Case 2, where the equilibrium X_3 is **unstable** and $X_1 = X_2 = 0$ is **stable**.

Case 4: $(\lambda_1, \lambda_2, \lambda_3) = (+, -, 0)$

We obtain a system of ODEs similar to that in Case 3, but with the condition that $\lambda_1 > 0$ and $\lambda_2 < 0$:

$$\begin{cases} \dot{X}_1 = \lambda_2 X_2 X_3 \\ \dot{X}_2 = -\lambda_1 X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases}, \text{ for } \lambda_1 > 0, \lambda_2 < 0.$$

Equilibrium solutions remain to be

$$X_1 = X_2 = 0 \quad \text{or} \quad X_2 = X_3 = 0 \quad \text{or} \quad X_3 = X_1 = 0.$$

The three relations can be obtained similarly

$$\begin{aligned} \lambda_1 X_1^2 + \lambda_2 X_2^2 &= c_1, \quad \text{and} \\ (\lambda_1 - \lambda_2) X_1^2 - \lambda_2 X_3^2 &= c_2, \quad \text{and} \\ (\lambda_1 - \lambda_2) X_2^2 + \lambda_1 X_3^2 &= c_3 \end{aligned}$$

However, with $\lambda_1 > 0, \lambda_2 < 0$, then $\lambda_1 - \lambda_2 > 0$. Hence, we have a hyperbolic cylinder around the equilibrium $X_1 = X_2 = 0$, so it is **unstable**. Around $X_3 = X_1 = 0$ and $X_2 = X_3 = 0$ are elliptic cylinders, so they are both **stable**.

Case 5: $(\lambda_1, \lambda_2, \lambda_3) = (+, +, +)$

The system of ODEs is

$$\begin{cases} \dot{X}_1 = (\lambda_2 - \lambda_3) X_2 X_3 \\ \dot{X}_2 = (\lambda_3 - \lambda_1) X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases}, \text{ for } \lambda_1, \lambda_2, \lambda_3 > 0.$$

Equilibrium solutions are given by

$$\begin{cases} (\lambda_2 - \lambda_3) X_2 X_3 = 0 \\ (\lambda_3 - \lambda_1) X_1 X_3 = 0 \\ (\lambda_1 - \lambda_2) X_1 X_2 = 0 \end{cases} \Rightarrow X_1 = X_2 = 0 \quad \text{or} \quad X_2 = X_3 = 0 \quad \text{or} \quad X_3 = X_1 = 0.$$

The three relations are

$$\begin{aligned} (\lambda_3 - \lambda_1) X_1^2 - (\lambda_2 - \lambda_3) X_2^2 &= c_1, \quad \text{and} \\ (\lambda_1 - \lambda_2) X_2^2 - (\lambda_3 - \lambda_1) X_3^2 &= c_2, \quad \text{and} \\ (\lambda_1 - \lambda_2) X_1^2 - (\lambda_2 - \lambda_3) X_3^2 &= c_3 \end{aligned}$$

- If $\lambda_1 = \lambda_2 = \lambda_3$, then all points are equilibria and are **stable**.
- If any two of the triple are equal, then the system degenerates back to Case 2, in which we have a **stable** equilibrium at the poles and **unstable** equilibria for the equator. For example, if $\lambda_1 = \lambda_2$, then the equilibria are $X_3 = 0$, which is **unstable** and $X_1 = X_2 = 0$, which is **stable**.
- If all of them are different, then we consider
  - If $\lambda_1 > \lambda_2 > \lambda_3 > 0$, then $\lambda_1 - \lambda_2 > 0$, $\lambda_2 - \lambda_3 > 0$ and $\lambda_3 - \lambda_1 < 0$. Hence around $X_1 = X_2 = 0$ $X_2 = X_3 = 0$ are elliptic cylinders, so they are both **stable**. Around $X_3 = X_1 = 0$ is a hyperbolic cylinder, so it is **unstable**.

- If $\lambda_2 > \lambda_3 > \lambda_1 > 0$, then $\lambda_2 - \lambda_3 > 0$, $\lambda_3 - \lambda_1 > 0$ and $\lambda_1 - \lambda_2 < 0$. Hence around $X_2 = X_3 = 0$ and $X_3 = X_1 = 0$ are elliptic cylinders, so they are both **stable**. Around $X_1 = X_2 = 0$ is a hyperbolic cylinder, so it is **unstable**.
- If $\lambda_3 > \lambda_1 > \lambda_2 > 0$, then $\lambda_3 - \lambda_1 > 0$, $\lambda_1 - \lambda_2 > 0$ and $\lambda_2 - \lambda_3 < 0$. Hence around $X_1 = X_2 = 0$ and $X_3 = X_1 = 0$ are elliptic cylinders, so they are both **stable**. Around $X_2 = X_3 = 0$ is a hyperbolic cylinder, so it is **unstable**.

Case 6: $(\lambda_1, \lambda_2, \lambda_3) = (+, +, -)$

We obtain a system of ODEs similar to that in Case 5, but with the condition that $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$:

$$\begin{cases} \dot{X}_1 = (\lambda_2 - \lambda_3) X_2 X_3 \\ \dot{X}_2 = (\lambda_3 - \lambda_1) X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases} \quad \text{, for } \lambda_1, \lambda_2 > 0, \lambda_3 < 0.$$

Equilibrium solutions remain to be

$$X_1 = X_2 = 0 \quad \text{or} \quad X_2 = X_3 = 0 \quad \text{or} \quad X_3 = X_1 = 0.$$

As $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$, $\lambda_2 - \lambda_3 > 0$ and $\lambda_3 - \lambda_1 < 0$, so we only need to compare λ_1 and λ_2.

- If $\lambda_1 > \lambda_2$, then $\lambda_1 - \lambda_2 > 0$. Hence around $X_1 = X_2 = 0$ and $X_2 = X_3 = 0$ are elliptic cylinders, so they are both **stable**. Around $X_1 = X_3 = 0$ is a hyperbolic cylinder, so it is **unstable**.
- If $\lambda_1 < \lambda_2$, then $\lambda_1 - \lambda_2 < 0$. Hence around $X_1 = X_2 = 0$ and $X_3 = X_1 = 0$ are elliptic cylinders, so they are both **stable**. Around $X_2 = X_3 = 0$ is a hyperbolic cylinder, so it is **unstable**.
- If $\lambda_1 = \lambda_2$, then $\lambda_1 - \lambda_2 = 0$, and the system degenerates back to Case 2, in which the equilibrium $X_1 = X_2 = 0$ is **stable** and the equilibrium $X_3 = 0$ is **unstable**.

5 Conclusion and Discussion

We have given a complete classification of stable and unstable geodesics for every 3-dimensional unimodular metric Lie algebra. The key finding can be summarised in the theorem below.

Theorem 5.1. *In the above notation, the equilibrium geodesics and their stability for three-dimensional unimodular Lie groups are given in the following table:*

$(\lambda_1, \lambda_2, \lambda_3)$		*Equilibrium*	*Stability*
$(0,0,0)$		*All points*	*Stable*
$(+,0,0)$		$X_2 = X_3 = 0$	*Stable*
		$X_1 = 0$	*Unstable*
$(+,+,0)$	$\lambda_1 = \lambda_2$	$X_1 = X_2 = 0$	*Stable*
		$X_3 = 0$	*Unstable*
	$\lambda_1 > \lambda_2$	$X_1 = X_2 = 0$	*Stable*
		$X_2 = X_3 = 0$	*Unstable*
		$X_3 = X_1 = 0$	*Stable*
	$\lambda_1 < \lambda_2$	$X_1 = X_2 = 0$	*Stable*
		$X_2 = X_3 = 0$	*Stable*
		$X_3 = X_1 = 0$	*Unstable*
$(+,-,0)$		$X_2 = X_3 = 0$	*Stable*
		$X_3 = X_1 = 0$	*Stable*
		$X_1 = X_2 = 0$	*Unstable*
$(+,+,+)$	$\lambda_1 = \lambda_2 = \lambda_3$	*All points*	*Stable*
	$\lambda_i \neq \lambda_k$	$X_i = X_j = 0$	*Stable*
		$X_k = 0$	*Unstable*
	$\lambda_i > \lambda_j > \lambda_k$	$X_i = X_j = 0$	*Stable*
		$X_j = X_k = 0$	*Stable*
		$X_k = X_i = 0$	*Unstable*
$(+,+,-)$	$\lambda_1 = \lambda_2$	$X_1 = X_2 = 0$	*Stable*
		$X_3 = 0$	*Unstable*
	$\lambda_1 > \lambda_2$	$X_1 = X_2 = 0$	*Stable*
		$X_2 = X_3 = 0$	*Stable*
		$X_3 = X_1 = 0$	*Unstable*
	$\lambda_1 < \lambda_2$	$X_1 = X_2 = 0$	*Stable*
		$X_2 = X_3 = 0$	*Unstable*
		$X_3 = X_1 = 0$	*Stable*

There are some very interesting remarks in the findings. For instance, the stability in Case 5 seems to follows a cyclic permutation, that is, if $\lambda_i > \lambda_j > \lambda_k$, then $X_i = X_j = 0$ and $X_j = X_k = 0$ are stable, while $X_k = X_i = 0$ is unstable. We again see that pattern in Case 6. One of the explanations can be from the nice symmetries that the unimodular Lie algebra possesses. One natural extension to this project is to look at the case when the 3-dimensional Lie algebra L is not unimodular, however, this requires a different approach to classify 3-dimensional non-unimodular Lie algebra (we can use Milnor's Lemma 3.1 [Milnor, 1976]). Under that condition, geodesics and their stability would need further investigation.

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