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Defining and Applying degrees for Linear Fredholm Operators

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Contents

Statement of Authorship

The idea behind this project came from Aidan (inspired by Iain's notes) and Glen (inspired by the lecture notes of Nečasová). While some of the more complex proofs are referenced and not done by Dion, the proofs of our degrees properties and equivalences among definitions are all Dion's work. Writing of the report and most of the research/source hunting was Dion's work as well.

Abstract

In this project we sought out to study the properties of Fredholm operators and then develop a degree theory for these operators. In order to do this we first looked into the Brouwer degree for operators on $\mathbb{R}^n$ and then the Leray-Schauder degree for operators in infinite dimensional Hilbert spaces of the form $1 - K$, where $K \in \mathcal{K}(\mathcal{H})$. Then by using the Leray-Schauder degree we could define and prove various properties of our Fredholm degree. Afterwards we outline potential uses of these degrees and discuss how they could be extended.

1 Introduction

Degree theory is greatly linked to the study of solution spaces of operators via constructing an integer-valued function that should (hopefully) remain invariant under small enough changes (continuous deformation, perturbation, etc). For finite dimensional operators (essentially operators on $\mathbb{R}^n$ and $\mathbb{C}^n$) a degree has been developed and studied quite in depth with the Brouwer degree and this study of solutions comes hand in hand with an important result from linear algebra; The Rank-Nullity Theorem.

A problem arises if we want to look at operators in infinite-dimensional spaces (for example l^2 or L^2) then we cannot depend on this result. This is where Fredholm operators come in as if we have an operator which is Fredholm with a non-negative index (ideally with index 0) then we get some form of rank-nullity back to let us better look at solutions of these operators. This is for the most part our motivation for developing a degree theory for Fredholm operators.

Before we get to a Fredholm degree however, we should develop intuition for degree theory by looking into the Brouwer degree and then provide a stepping stone by studying the Leray-Scahuder degree.

1.1 Linear Fredholm operators

Before we can work with degrees of Fredholm operators, we must also first define the operators in question and find some of their properties.

Definition 1.1.1. Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces and let $L \in \mathcal{L}(\mathcal{H}, \mathcal{G})$. We say L is *Fredholm* if it has finite dimensional kernel and finite dimensional cokernel. If L is Fredholm we define the *Fredholm index of* L by:

$$\mathrm{ind}(L) := \dim \ker(L) - \dim \mathrm{coker}(L)$$

Where $\mathrm{coker}(L) = \mathcal{G}/L(\mathcal{H}) \cong L(\mathcal{H})^{\perp} \cong \ker(L^*)$. We denote the set of Fredholm operators by $\mathcal{F}(\mathcal{H}, \mathcal{G})$.

Remark 1.1.2. *Every operator $L \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is Fredholm with index $m - n$.*

Definition 1.1.3 (Properties of Fredholm operators)**.** Let $S, L \in \mathcal{F}(\mathcal{H}, \mathcal{G})$ and $K \in \mathcal{K}(\mathcal{H}, \mathcal{G})$. Then:

(a) $\exists B \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ such that $1 - LB$ and $1 - BL$ are finite rank projections onto $L(\mathcal{H})^{\perp}$ and $\ker(L)$ respectively.

(b) $L^* \in \mathcal{F}(\mathcal{H}, \mathcal{G})$ and $\mathrm{ind}(L^*) = -\mathrm{ind}(L)$.

(c) $SL \in \mathcal{F}(\mathcal{H}, \mathcal{G})$ and $\mathrm{ind}(SL) = \mathrm{ind}(S) + \mathrm{ind}(L)$.

(d) $L - K \in \mathcal{F}(\mathcal{H}, \mathcal{G})$ and $\mathrm{ind}(L - K) = \mathrm{ind}(L)$; in particular, we have $\mathrm{ind}(1 - K) = 0$ (A.K.A the Fredholm Alternative).

Lemma 1.1.4. *Let $L \in \mathcal{F}_0(\mathcal{H}, \mathcal{G})$. Then $\exists N \in GL(L(\mathcal{H}), \mathcal{H})$ such that*

$$NL = 1 - K$$

for some $K \in \mathcal{K}(\mathcal{H})$.

Proof. By Definition 1.1.3 (a) we know $\exists N \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ such that $NL = 1 - K$, where $K = \mathrm{proj}_{\ker(L)} \in \mathrm{FR}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$. Also as L and $1 - K$ have index 0,

$$\mathrm{ind}(N) = \mathrm{ind}(N) + \mathrm{ind}(L) = \mathrm{ind}(NL) = \mathrm{ind}(1 - K) = 0.$$

Let $y_1, y_2 \in L(\mathcal{H})$ (so $y_1 = Lx_1$ and $y_2 = Lx_2$) such that $Ny_1 = Ny_2$. Then:

$$N(y_1 - y_2) = 0$$
$$NL(x_1 - x_2) = 0$$
$$\text{proj}_{\ker(L)^\perp}(x_1 - x_2) = 0$$

As $\ker(L)$ is closed, we can write $\mathcal{H} = \ker(L) \oplus \ker(L)^\perp$ and as $\text{proj}_{\ker(L)^\perp}(x_1 - x_2) = 0$, $x_1 - x_2 \in \ker(L) \implies L(x_1) = L(x_2) \implies y_1 = y_2$ and thus N is injective on $L(\mathcal{H})$.

Since N is injective, $\dim\ker(L) = 0$ and as N has index 0, $\dim\text{coker}(L) = 0$ which implies that $\mathcal{H}/N(L(\mathcal{H})) = \{[0]\}$. So $\forall x \in \mathcal{H}$, $\exists y \in NL(\mathcal{H})$ such that $x = 0 + y = y$, making $N(L(\mathcal{H})) = \mathcal{H}$ and N surjective on $L(\mathcal{H})$. ☕

Corollary 1.1.5. *We can extend Lemma 1.1.4 to a path of operators $L_t \in \mathcal{F}_0(\mathcal{H}, \mathcal{G})$ for each $t \in [a, b]$ so that there exists a path $N_t \in GL(\mathcal{G}, \mathcal{H})$ in which $N_tL_t = 1 - K_t$ for some path of compact operators K_t. The path N_t need not be unique.*

1.2 Degree for Finite-Dimensional Operators

To see the motivation behind our definition for the degree of Fredholm operators, it's best to first look at a degree that's defined in the finite-dimensional case.

Definition 1.2.1 (The Brouwer Degree)**.** Let $F \in \mathcal{L}(\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ be an open subset such that $F : \overline{\Omega} \to \mathbb{R}^n$ is continuous (equivalent to bounded for linear operators) and proper ($\Gamma \in \mathbb{R}^n$ compact $\implies$ $F^{-1}(\Gamma)$ compact), and $y \in \mathbb{R}^n \backslash F(\partial\Omega)$ be fixed. We define the Brouwer degree of F on Ω at y to be:

$$d_B(F, \Omega, y) := \begin{cases} 0, & y \notin F(\Omega) \\ \sum_{x \in F^{-1}(y)} \text{sign}\det(DF_x), & y \in F(\Omega) \end{cases}$$
$$= \begin{cases} 0, & y \notin F(\Omega) \\ \text{sign}\det(F), & y \in F(\Omega) \end{cases}$$

The former being the general definition which may require more conditions to be met in order to be defined and the latter being the case for purely linear operators. This definition came from the lecture notes for PDE applications [3].

Definition 1.2.2. Let F, Ω, and y be as in Definition 1.2.1. Then:

(a) If Ω_1, Ω_2,..., Ω_n are open subsets of Ω such that $\bigcup_{i=1}^n \overline{\Omega_i} = \overline{\Omega}$ and $y \notin \bigcup_{i=1}^n F(\partial\Omega_i)$, then $d_B(F, \Omega, y) = \sum_{i=1}^n d_B(F, \Omega_i, y)$.

(b) If we have an open subset $\Omega' \subset \Omega$ such that $y \in F(\Omega')$ then $d_B(F, \Omega', y) = d_B(F, \Omega, y)$.

(c) For all $b \in \mathbb{R}^n$, $d_B(F, \Omega, y) = d_B(F_b, \Omega, y - b)$, where $F_b(x) = F(x) - b$.

(d) If we define a continuous map $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ such that $H(t, \cdot)$ is continuous and proper then for $y \notin H([0, 1] \times \partial\Omega)$, $d_B(H(t_1, \cdot), \Omega, y) = d_B(H(t_2, \cdot), \Omega, y)$ for all $t_1, t_2 \in [0, 1]$.

This degree is rather simple to construct and use, however we can only use it for finite dimensional operators as in an infinite-dimensional space we lose the notion of a determinant. In order to have a degree for those spaces we must find a property that is 'equivalent' to the sign of a determinant.

2 The Leray-Schauder Degree

2.1 Construction

Definition 2.1.1. Let $\mathcal{H}$ be a Hilbert space and let $L \in \mathcal{L}(\mathcal{H})$. We say L is *linearly Leray-Schauder (or L.S)* if we can write $L = 1 - K$ for some $K \in \mathcal{K}(\mathcal{H})$. We denote the space of all L.S operators $\mathcal{S}(\mathcal{H})$.

We should note that $\mathcal{S}(\mathcal{H}) \subset \mathcal{F}_0(\mathcal{H})$ as a result of the Fredholm Alternative.

Definition 2.1.2. Let L_0 and L_1 be L.S operators. We say L_0 and L_1 are *L.S Homotopic* if there exists a continuous path $P : [0,1] \times \mathcal{H} \to \mathcal{H}$ such that

(a) $P(t, \cdot)$ is L.S for every $t \in [0,1]$, and

(b) $P(0,x) = L_0(x)$ and $P(1,x) = L_1(x)$

For some L.S operator L, we can define a homotopy class H_L by all operators that are L.S homotopic to L.

Theorem 2.1.3 (The Linear Homotopy Theorem). *If we define H^+ by the homotopy class containing the identity map and H^- by the homotopy class containing the map 1^- defined by $(x_1, x_2, \ldots) \mapsto (-x_1, x_2, \ldots)$ (both of which are L.S), then every L.S operator is either in H^+ or H^-.*

The proof of this theorem is beyond the scope of what is covered in this project, but the main idea is to use the index that will be defined below and show that it is constant for any continuous path. The full proof can be found in the appendix of Rothe [2].

Definition 2.1.4. We define the index function $j : \mathcal{S}(\mathcal{H}) \to \{1, -1\}$ by

$$j(L) := \begin{cases} 1, & L \in H^+ \\ -1, & L \in H^- \end{cases}$$

Remark 2.1.5. *If $\mathcal{H}$ is finite-dimensional and real-valued, then we can place an operator $L \in \mathcal{S}(\mathcal{H})$ into one of these homotopy classes by:*

$$L \in \begin{cases} H^+, & \det(L) > 0 \\ H^-, & \det(L) < 0 \end{cases}$$

Using this idea of placing an operator into one of two homotopy classes via theorem 2.1.3 gives us back the sense of orientation that we lose by studying operators in infinite dimensional spaces. From this we can now define:

Definition 2.1.6 (The Leray-Schauder Degree)**.** Let Ω be an open subset of $\mathcal{H}$ and let $L \in \mathcal{S}(\overline{\Omega}, \mathcal{H})$ be proper. For $y \in \mathcal{H} \backslash L(\partial\Omega)$ we define the *Leray-Schauder degree of L on Ω at y* by:

$$\begin{aligned} d_{L.S}(L, \Omega, y) :&= \begin{cases} 0, & y \notin L(\Omega) \\ \sum_{x \in L^{-1}(y)} j(DL_x), & y \in L(\Omega) \end{cases} \\ &= \begin{cases} 0, & y \notin L(\Omega) \\ j(L), & y \in L(\Omega) \end{cases} \end{aligned}$$

and just like the Brouwer degree, the former definition is for general operators of the form $L = 1 - K$ and the latter is the case when L is linear.

2.2 Properties

By construction of the index j, we immediately attain L.S homotopy invariance of the degree. That is, if $H(t, x)$ is a continuous map connecting two L.S maps L_0 and L_1 such that $H(t, \cdot)$ is a L.S map for all $t \in [0, 1]$ then $d_{L.S}(L_0, \Omega, y) = d_{L.S}(L_1, \Omega, y)$. But we can show other forms of invariance that this degree has.

Proposition 2.2.1. If Ω_1, Ω_2,…, Ω_n are open subsets of Ω such that $\bigcup_{i=1}^n \overline{\Omega_i} = \overline{\Omega}$ and $y \notin \bigcup_{i=1}^n L(\partial\Omega_i)$, then $d_{L.S}(L, \Omega, y) = \sum_{i=1}^n d_{L.S}(L, \Omega_i, y)$.

Proof. Suppose $y \notin L(\Omega)$. Then $y \notin \bigcup_{i=1}^n L(\Omega_i)$ and $d_{L.S}(L, \Omega_i, y) = 0$, $\forall i$.

Now suppose $y \in L(\Omega)$. Then since each Ω_i is disjoint, $y \in L(\Omega_j)$ for exactly one $j \in \{1, 2, \dots, n\}$. So

$$\sum_{i=1}^{n} d_{L.S}(L, \Omega_i, y) = d_{L.S}(L, \Omega_j, y) = j(L) = d_{L.S}(d, \Omega, y)$$

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Corollary 2.2.2. *If Ω' is an open subset of Ω such that $y \in L(\Omega')$, then $d_{L.S}(L, \Omega, y) = d_{L.S}(L, \Omega', y)$.*

Proposition 2.2.3. For all $b \in \mathcal{H}$, $d_{L.S}(L_b, \Omega, y) = d_{L.S}(L, \Omega, y - b)$, where $L_b(x) = L(x) + b$.

Proof. As L_b is an affine operator (not linear), we must use the more general definition of our degree:

$$\begin{aligned}
d_{L.S}(L_b, \Omega, y) &= \begin{cases} 0, & y \notin L_b(\Omega) \\ j(DL_{b,x}), & y \in L_b(\Omega) \end{cases} \\
&= \begin{cases} 0, & y \notin \{L(\Omega) + b\} \\ j(D(L(x) + b)), & y \in \{L(\Omega) + b\} \end{cases} \\
&= \begin{cases} 0, & y - b \notin L(\Omega) \\ j(L), & y - b \in L(\Omega) \end{cases} \\
&= d_{L.S}(L, \Omega, y - b)
\end{aligned}$$

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Proposition 2.2.4. Let $L, M \in \mathcal{S}(\mathcal{H})$. Then $d_{L.S}(L \circ M, \Omega, y) = d_{L.S}(M, \Omega, y) \cdot d_{L.S}(L, M(\Omega), My)$.

Proof. The degree of $L \circ M$ is nonzero iff $y \in (L \circ M)(\Omega)$ which implies that $y \in M(\Omega)$ and $My \in L(M(\Omega))$. So $d_{L.S}(L \circ M, \Omega, y) \neq 0$ iff $d_{L.S}(M, \Omega, y) \cdot d_{L.S}(L, M(\Omega), My) \neq 0$.

<u>Case 1: $L, M \in H^+$.</u> Then as there is a continuous map H connecting M and 1, the map $L \circ H$ connects $L \circ M$ and $L \circ 1 = L$ which can then continuously be deformed to 1, so

$L \circ M \in H^+$. That is:

$$d_{L.S}(L \circ M, \Omega, y) = 1 = 1 \cdot 1 = d_{L.S}(M, \Omega, y) \cdot d_{L.S}(L, M(\Omega), My).$$

Case 2: $L, M \in H^-$. By a similar argument to case 1, for a continuous map H connecting M to 1^-, $L \circ H$ connects $L \circ M$ to $L \circ 1^-$. But as L can be continuously deformed to 1^-, $L \circ 1^-$ can be deformed to $1^- \circ 1^- = 1$. That is:

$$d_{L.S}(L \circ M, \Omega, y) = 1 = -1 \cdot -1 = d_{L.S}(M, \Omega, y) \cdot d_{L.S}(L, M(\Omega), My).$$

From the above two cases we can see that elements in H^+ preserve "orientation" when composed with other L.S operators and elements in H^- switch "orientation". So if $L \in H^+$ and $M \in H^-$ (or vice-versa) then since orientation is switched only once, $L \circ M \in H^-$ and

$$d_{L.S}(L \circ M, \Omega, y) = -1 = -1 \cdot 1 = d_{L.S}(M, \Omega, y) \cdot d_{L.S}(L, M(\Omega), My).$$

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3 Fredholm degree

While the Leray-Schauder degree is a well-behaving function that works for infinite dimensional operators, it can only be used for operators of the form $L = 1 - K$, which is only a small subset of all Fredholm operators of index 0. In order to extend this degree, we need some more powerful tools under our belt.

3.1 Construction

Definition 3.1.1. Let L_t, N_t, and K_t be as they are in corollary 1.1.5 for some fixed interval $I = [a, b]$. Furthermore suppose that L_a & L_b are invertible. Then we define the parametrix invariant, homotopy invariant function $\sigma : \mathcal{F}_0(\mathcal{H}, \mathcal{G}) \times I \to \{1, -1\}$ by

$$\sigma(L, I) := d_{L.S}(N_a L_a, B_\varepsilon(0), 0) \cdot d_{L.S}(N_b L_b, B_\varepsilon(0), 0).$$

We call σ *the parity of L on I*. We have this definition from the Fitzpatrick paper [4].

Remark 3.1.2. *We can define the parity using degrees at 0 as Proposition 2.2.3 allows us to assume WLOG that any L.S operator has a solution at 0.*

To show that σ is parametrix invariant, Let $N_{1,t}, N_{2,t} \in \mathrm{GL}(\mathcal{G}, \mathcal{H})$.

Case 1: Suppose that $H(\cdot, \tau)$ is a continuous path between $N_1 L$ and $N_2 L$ such that it is L.S for all τ. Then $N_{1,a} L_a$ is L.S homotopic to $N_{2,a} L_a$ & $N_{1,b} L_b$ is L.S homotopic to $N_{2,b} L_b$, meaning that $d_{L.S}(N_{1,a} L_a, B_\varepsilon(0), 0) = d_{L.S}(N_{2,a} L_a, B_\varepsilon(0), 0)$ and $d_{L.S}(N_{1,b} L_b, B_\varepsilon(0), 0) = d_{L.S}(N_{2,b} L_b, B_\varepsilon(0), 0)$.

Case 2: Now suppose that there are no continuous paths connecting $N_1 L$ to $N_2 L$ while staying L.S on the entire path. So $d_{L.S}(N_{1,a} L_a, B_\varepsilon(0), 0) = -d_{L.S}(N_{2,a} L_a, B_\varepsilon(0), 0)$ and $d_{L.S}(N_{1,b} L_b, B_\varepsilon(0), 0) = -d_{L.S}(N_{2,b} L_b, B_\varepsilon(0), 0)$, meaning that

$$\begin{aligned}\sigma(L, I) &= d_{L.S}(N_{1,a} L_a, B_\varepsilon(0), 0) \cdot d_{L.S}(N_{1,b} L_b, B_\varepsilon(0), 0) \\ &= [-d_{L.S}(N_{2,a} L_a, B_\varepsilon(0), 0)] \cdot [-d_{L.S}(N_{2,b} L_b, B_\varepsilon(0), 0)] \\ &= d_{L.S}(N_{2,a} L_a, B_\varepsilon(0), 0) \cdot d_{L.S}(N_{2,b} L_b, B_\varepsilon(0), 0)\end{aligned}$$

So in either case, we end up with the same value for the parity. To show homotopy invariance just involves using parametrix invariance alongside the fact that the L.S degree is homotopy invariant as well.

Definition 3.1.3 (The Fredholm Degree)**.** Let $\mathcal{O} \subset \mathcal{H}$ be an open and simply connected set such that $0 \in \mathcal{O}$ and let $L \in \mathcal{F}_0(\mathcal{O}, \mathcal{H})$. Also let $\Omega \subset \mathcal{O}$ be any open subset such that $L|_{\overline{\Omega}}$ is proper.

Fix $p \in \mathcal{O}$ such that $L(p)$ is invertible (which we will call the *base point*) and $y \in \mathcal{H} \backslash L(\partial\Omega)$. If $L^{-1}(y)$ is nonempty then there exists a set of k continuous paths $\gamma_j : [0,1] \to \mathcal{O}$ connecting p to the solutions $x_j \in L^{-1}(y)$.

We define *the degree of* L *on* Ω *at* y *with base point* p by

$$d_p(L, \Omega, y) := \begin{cases} 0, & y \notin L(\Omega) \\ \sum_{j=1}^{k} \sigma(DL_{x_j} \circ \gamma_j, [0,1]), & y \in L(\Omega) \end{cases}$$
$$= \begin{cases} 0, & y \notin L(\Omega) \\ \sigma(L \circ \gamma, [0,1]), & y \in L(\Omega) \end{cases}$$

If we focus on the linear case, by construction $L(p)$ is invertible and since x is a solution to $L(x) = y$ we know $L(x)$ is invertible, so $\sigma(L \circ \gamma, [0,1])$ is well defined. Moreover, if we pick an alternate path γ' connecting p to x then since the parity is only defined by the endpoints of the path, $\sigma(L \circ \gamma, [0,1]) = \sigma(L, \circ\gamma', [0,1])$.

3.2 Properties

Proposition 3.2.1. let q be another base point for L and let $\tau : [0,1] \to \mathcal{O}$ be a path connecting q and p. Then $d_q(L, \Omega, y) = \sigma(L \circ \tau, [0,1]) \cdot d_p(L, \Omega, y)$.

Proof. First, suppose that $y \notin L(\Omega)$. Then $d_p(L, \Omega, y) = 0$ for any base point p. Otherwise:

$$\begin{aligned} d_q(L, \Omega, y) &= \sigma(L \circ (\gamma \circ \tau), I) \\ &= d_{L.S}(NL, \Omega, L(q)) \cdot d_{L.S}(NL, \Omega, y) \\ &= d_{L.S}(NL, \Omega, L(q)) \cdot 1 \cdot d_{L.S}(NL, \Omega, y) \\ &= [d_{L.S}(NL, \Omega, L(q)) \cdot d_{L.S}(NL, \Omega, L(p))] \cdot [d_{L.S}(NL, \Omega, L(p)) \cdot d_{L.S}(NL, \Omega, y)] \\ &= \sigma(L \circ \tau, I) \cdot d_p(L, \Omega, y) \end{aligned}$$

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Remark 3.2.2. *From the above proposition we can see that by extending this degree to all Fredholm operators we lose the homotopy invariance that we have with the L.S degree. This is due to the fact that by continuously deforming one operator into another we also change the base point (if the base point stayed constant throughout the transformation then we would have homotopy invariance).*

Every other property stated earlier for the L.S degree remains true for the Fredholm degree as they do not involve a change of base point.

4 Uses/Applications

4.1 Existence of solutions for the integral equation

Fix $I = [a, b]$ and $k \in C^0(I^2)$ linear. For some $h \in \mathrm{L}^2(I)$ we define the *integral operator with kernel k* $K_k h : I \to \mathbb{C}$ by

$$(K_k h)(s) := \int_a^b k(s,t)h(t)\,dt$$

Because $[a, b]$ has finite measure, Hölder's inequality implies that h is integrable so for each s the integral makes sense.

Claim. $K_k h \in \mathcal{L}(\mathrm{L}^2(I))$.

Proof. Since k is linear, $(K_k h)(s)$ is linear. To get boundedness, it suffices to show continuity, so fix $\varepsilon > 0$ and let $s, t \in I$. Then:

$$\begin{aligned} ||(K_k h)(s) - (K_k h)(t)||_2 &= \left|\left|\int_a^b k(s,\tau)h(\tau) - k(t,\tau)h(\tau)\,d\tau\right|\right|_2 \\ &\le \left|\left|\int_a^b ||k(s,\tau) - k(t,\tau)||_\infty h(\tau)\,d\tau\right|\right|_2 \\ &= ||k(s,\tau) - k(t,\tau)||_\infty \left|\left|\int_a^b h(\tau)\,d\tau\right|\right|_2 \end{aligned}$$

As k is continuous and h is integrable, $\exists\delta > 0$ such that for $|s - t| < \delta$ we have $|k(s,\tau) - k(t,\tau)| < \varepsilon / \left|\left|\int_a^b h(\tau)\,d\tau\right|\right|_2$ for all $\tau \in I$ and thus $||k(s,\tau) - k(t,\tau)||_\infty < \varepsilon / \left|\left|\int_a^b h(\tau)\,d\tau\right|\right|_2$. Therefore for $|s - t| < \delta$:

$$||(K_k h)(s) - (K_k h)(t)||_2 < \frac{\varepsilon}{\left|\left|\int_a^b h(\tau)\,d\tau\right|\right|_2} \cdot \left|\left|\int_a^b h(\tau)\,d\tau\right|\right|_2 = \varepsilon$$

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From here it can be proven that $K_k h \in \mathcal{K}(\mathrm{L}^2(I))$, and while the proof is quite complex and requiring some knowledge in *-algebras, the basic idea of it is to use operators of the form $f(s)g(t)$ for $f, g \in \mathrm{C}^0(I)$ to approximate k. A more laid out proof of this can be found in Iain's notes [1]. Since $K_k h$ is compact, the operator

$$(1 + K_k h)(s) = h(s) + \int_a^b k(s,t)h(t)\,dt$$

is L.S. We call this the *integral equation* and proving existence of a solution to the equation $(1 + K_k h)(s) = f(s)$ for some function $f \in (1 + K_k h)(\Omega) \subset \mathrm{L}^2(I)$ can be done by showing that $d_{L.S}(1 + K_k h, \Omega, f) \neq 0$. However, by proposition 2.2.3, to prove existence of a solution for any f it is enough to show that $d_{L.S}(1 + K_k h, \Omega, 0) \neq 0$ which is equivalent to showing that the homogeneous integral equation

$$h(s) + \int_a^b k(s,t)h(t)\,dt = 0$$

has nonzero $\mathrm{L}^2(I)$ solutions.

4.2 Global Bifurcation

Fix a Hilbert space $\mathcal{H}$ and let $F \in \mathcal{F}_1(\mathbb{R} \times \mathcal{H}, \mathcal{H})$ (so that for each $\lambda \in \mathbb{R}$, $F(\lambda, \cdot) \in \mathcal{F}_0(\mathcal{H})$) and suppose that $F(\lambda, 0) = 0, \forall \lambda \in \mathbb{R}$ (for example, $F(\lambda, x) = x - \lambda K x$ for some $K \in \mathcal{K}(\mathcal{H})$).

Suppose $\exists \lambda_-, \lambda_+ \in \mathbb{R}$ with $\lambda_- < \lambda_+$ such that $F(\lambda_\pm, 0)$ is invertible and that

$$\sigma(F(\cdot, 0), [\lambda_-, \lambda_+]) = -1.$$

If S denotes the closure in $\mathbb{R} \times \mathcal{H}$ of the nontrivial solutions of $F(\lambda, x) = 0$ and C is the connected component of $\mathrm{S} \cup [\lambda_-, \lambda_+] \times \{0\}$ containing $[\lambda_-, \lambda_+] \times \{0\}$ then either

(a) C is not compact, or

(b) C contains a point $(\lambda_1, 0)$ with $\lambda_1 \notin [\lambda_-, \lambda_+]$.

The proof of this, while requiring a fair bit of groundwork, can be found in Pejsachowicz-Rabier [6]. The reason this method stands out from other ways to find bifurcation is because it's the only way that we get global bifurcation that is invariant under homotopy.

5 Discussion/Conclusion

Given more time (or perhaps more direction), we could look into one of the applications specified above in more depth, however an alternate route for continuation could be trying to further extend the degree. This could be done by extending the properties of Fredholm operators to Banach spaces which allows us to extend these degrees for use in L^p spaces or even $W^{k,p}$ spaces. We could also work on defining these degrees for nonlinear operators given some level of smoothness and then using approximations of those smoother functions to define degrees for less smooth functions.

6 Appendix

6.1 Alternative definitions

Definition 6.1.1 (Alternate Brouwer degree)**.** Fix $\mathcal{H} = \mathbb{R}^n$ (or $\mathbb{C}^n$) and let $F \in \mathcal{L}(\mathcal{H})$, $\Omega \in \mathcal{H}$, $y \in \mathcal{H}\backslash F(\partial\Omega)$. Let $\Phi(= \Phi(r)) : [0, \infty] \to \mathbb{R}$ be a continuous function that vanishes on a neighbourhood around 0 and for $\varepsilon < r < \infty$, where $0 < \varepsilon < \min_{x\in\partial\Omega}(||F(x) - y||)$ and

$$\int_{\mathcal{H}} \Phi(||x||)\, dx = 1.$$

We define the *Brouwer dergee of F on Ω at y* to be:

$$d_B(F, \Omega, y) := \int_{\Omega} \Phi(||F(x) - y||) J[F(x)]\, dx$$

and is equivalent to definition 1.2.1 when $\mathcal{H}$ is real-valued. We get this defintion from Heinz [5]. Note that this definition is independent of our function Φ.

Proof of equivalence. While it can be proven for general operators, we will only show that the definitions for linear operators are equivalent. First, suppose that $y \notin F(\Omega)$. Then for any $x \in \Omega$, $||F(x) - y|| > \varepsilon$ and $\Phi(||F(x) - y||) = 0$ so $d_B(F, \Omega, y) = 0$, agreeing with our original definition.

Now suppose $y \in F(\Omega)$. Then for $x_0 = F^{-1}(y)$:

$$\begin{aligned} d_B(F,\Omega,y) &= \int_\Omega \Phi(||F(x)-y||)J(F(x))\,dx \\ &= \int_\Omega \Phi(||F(x)-y||)\det(F)\,dx \\ &= \det(F)\int_{B_\varepsilon(x_0)} \Phi(||F(x-x_0)||)\,dx \\ &= \det(F)\int_{F^{-1}(B_\varepsilon(0))} \frac{\Phi(||u||)}{|\det(F)|}\,du \\ &= \frac{\det(F)}{|\det(F)|}\cdot 1 \\ &= \begin{cases} 1, & \det(F)>0 \\ -1, & \det(F)<0 \end{cases} \end{aligned}$$

Which is equivalent to our original definition. ☕

Remark 6.1.2. *Using this alternate definition it is much easier to prove certain properties, such as homotopy invariance (the proof of which can be found in [5]).*

Definition 6.1.3 (Alternate L.S degree for linear operators)**.** Let $L = 1 - K \in \mathcal{S}(\mathcal{H})$ and $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of K with real part greater than 1. If we let m be the sum of algebraic multiplicities of each λ_i then we can define the Leray-Schauder degree of L on $\Omega \subset \mathcal{H}$ at $y \in \mathcal{H}\backslash L(\partial\Omega)$ to be:

$$d_{L.S}(L,\Omega,y) := (-1)^m$$

and is equivalent to definition 2.1.6. This definition can be found in the PDE lecture notes [3].

This is actually the original definition of the L.S degree which can be shown to be equivalent to our original definition using a lot of groundwork which isn't covered here as it requires a lot of spectral theory. The proof of equivalence can be found in the Appendix of [2].

6.2 Outline of Extension to nonlinear operators

Definition 6.2.1 (Brouwer degree for nonlinear operators)**.** Let $\Omega \subset \mathbb{R}^n$ be an open + bounded subset and $F \in C^1(\Omega,\mathbb{R}^n)$ such that $F|_{\overline{\Omega}}$ is continuous and proper. Fix $y \in \mathbb{R}^n\backslash F(\partial\Omega)$ such that y is a regular value (So DF_x is invertible for all $x \in F^{-1}(y)$). Since F is proper and $y \notin F(\partial\Omega)$, $F^{-1}(y)$ is a finite subset of Ω which allows us to define the *Brouwer degree of F on Ω at y* to be:

$$d_B(F, \Omega, y) = \sum_{x \in F^{-1}(y)} \operatorname{sign} \det(DF_x)$$

Definition 6.2.2 (L.S degree for nonlinear operators)**.** Fix an open subset $\Omega \in \mathbb{R}^n$, a compact operator $K \in C^1(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$ and let $L = 1 - K$. then for a regular value $y \in \mathbb{R}^n \backslash L(\partial\Omega)$ we define *the L.S degree of L on Ω at y* to be

$$d_{L.S}(L, \Omega, y) = \sum_{x \in L^{-1}(y)} j(DL_x)$$

Where $j(\cdot)$ is our index from definition 2.1.4.

Most properties that we have for the linear case extend fairly easily to the nonlinear case, however in order to get homotopy invariance for the above degrees we need to ensure that our operator is in C^2. This allows us to use Sard's theorem (or Sard-Smale in infinite dimensional spaces) which implies that the set of critical values of an operator given any set in the domain is a set of Lebesgue measure 0. Not only does this let us get around those critical points to create a homotopy path, but it also lets us define degrees at those critical points by looking at degrees for close enough regular points.

Then for the Brouwer degree in particular, by using the Weierstrass approximation theorem, for any operator $L \in C^0(\overline{\Omega}, \mathbb{R}^n)$ we can find some operator $L' \in C^2(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$ such that for all $\varepsilon > 0$, $||Lx - L'x|| < \varepsilon$ and thus for any $y \notin L(\partial\Omega)$, we have the equality $d_B(L, \Omega, y) = d_B(L', \Omega, y)$. We can achieve this property for the L.S degree but it requires some more rigorous work (See chapter 3 of [2]).

The former definition given for our Fredholm degree in definition 3.1.3 is essentially the general case, if we assume that $L \in C^1(\Omega, \mathcal{H}) \cap C^0(\overline{\Omega}, \mathcal{H})$ and Fredholm of index 0. While we still have the same homotopy variance that we get from the linear case, for the nonlinear case it makes more sense to define a homotopy invariant 'absolute' degree independent of base point p:

$$|d|(L, \Omega, y) := |d_p(L, \Omega, y)|.$$

References

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