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Order Unit Equivalences and Structural Properties of Leavitt Path Algebras

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1 Abstract

There is an intimate connection between the geometry of a graph and its associated Leavitt path algebra and monoid. In this note we explore these connections by investigating order unit structures within the graded monoid. We demonstrate a simple proof on the free finitely generated graded-projective modules of $L_K(E)$. We go on to introduce several order unit equivalence criteria, showing how these criteria may capture further information on the algebraic structure of the associated Leavitt path algebras. These all suggest that the talented monoid with its associated grading may be a crucial tool in finding a complete invariant for these algebras.

2 Introduction

When considering the classification of graphs we may study how the geometry of the graph corresponds to an associated algebra and additionally contemplate whether there exists a complete invariant for the the classification of these algebras. The so called *Leavitt Algebra*, $L_K(1, n)$ corresponding to an integer n over a field K, was defined in the 1960's by W.G. Leavitt, being the universal class of algebras that fail to have the *Invariant Basis Number* property. The recent identification of a graph criterion for when the associated algebra is simple in 2005 [1] alongside the description of the non-stable K-theory of these algebras [3] by a natural monoid associated to their graphs has flourished into the study of the so called *Leavitt path algebras*. In this paper we expand upon recent work by R. Hazrat and H. Li [5] investigating the correspondence between the geometry of the graph and the *graded monoid* of the associated Leavitt path algebra.

Let E be a row-finite directed graph with vertices E^0 and edges E^1. The *talented* monoid is defined as the free abelian monoid over the vertices E^0 subject to identifying a vertex with the sum of vertices it connects to, and indexing the vertices with a $\mathbb{Z}$-action keeping track of the transformations:

$$T_E = \langle v(i), v \in E^0, i \in \mathbb{Z} \mid v(i) = \textstyle\sum_{v \to u} u(i+1) \rangle.$$

The action of $n \in \mathbb{Z}$ on $v(i)$ is defined by $v(i+n)$ and denoted ${}^n v(i)$. In [5], Hazrat and Li note that there are $\mathbb{Z}$-module isomorphisms:

$$T_E \cong M_{\overline{E}} \cong \mathcal{V}(L_F(\overline{E})) \cong \mathcal{V}^{gr}(L_F(E))$$

$$v(i) \mapsto v_i \mapsto [L_F(\overline{E})v_i] \mapsto [(L_F(E)v)(i)]$$

Where $\mathcal{V}^{gr}(L_F(E))$ is the monoid of graded finitely generated projective modules of the Leavitt path algebra $L_F(E)$. The group completion of T_E then retrieves the *graded Grothendiek group* $K_0^{gr}(L_F(E))$. The non-graded version of this group has been of significant use in the classification of C^*-algebras and graph C^*-algebras that may be considered the analytic counterpart to Leavitt path algebras. We find that the action of $\mathbb{Z}$ on T_E corresponds to the shift operations on graded modules over the Leavitt path algebra $L_F(E)$ which is naturally $\mathbb{Z}$ graded.

In [4] Hazrat identifies a criterion for the graded isomorphism of Leavitt Path Algebras:

There is a graded module isomorphism $L_n^k(\lambda_1,...,\lambda_k) \cong_{gr} L_n^{k'}(\gamma_1,...,\gamma_{k'})$ if and only if $\sum_{i=1}^k n^{\lambda_i} = \sum_{i=1}^{k'} n^{\gamma_i}$.

For a finite directed graph $E, \lambda \in \mathbb{Z}$, we denote by $1_E(\lambda)$ the element $\sum_{u\in E^0} u(\lambda) \in T_E$. We say that x is an *order unit* if for any $a \in T_E$ we have $a \le \sum_{i=1}^k x(\lambda_i)$, where $\lambda_i \in \mathbb{Z}$. It is clear to see that 1_E is an order unit and that we have $\phi(1_E(\lambda_i)) = [L(E)\lambda_i]$.

Our aim is to identify order unit equivalence criteria within the talented monoid of associated graphs to illuminate the connections between the geometry of a graph E, the associated T_E monoid and the corresponding algebraic structure of $L_F(E)$.

We define an order unit equivalence for when the corresponding $L_F(E)$ algebra has the graded non-invariant basis number property as well as defining an additional two novel criteria under which to consider order unit equivalence within T_E. We begin to analyse graph geometries attempting to identify conditions under which these criteria hold, including some incomplete proofs as demonstration of the techniques involved. We go on to prove a property concerning the graded-projective $L(E)$-modules to conclude the note.

The additional T_E structure and relations to $L_F(E)$ identified within this paper provide further evidence that the talented monoid and its associated grading contain crucial information when considering a complete invariant for these algebras. This is of import due to the graded version of the algebraic Hirchsberg-Phillips conjecture. It was conjectured in [4], that the graded Grothendiek group K_0^{gr} along with its ordering and module structure is a complete invariant for the class of finite Leavitt path algebras:

Conjecture 1. *Let E_1 and E_2 be finite graphs and F a field. Then the following are equivalent:*

1. *There is a $\mathbb{Z}$-module isomorphism $\phi : T_{E_1} \to T_{E_2}$, such that $\phi(\sum_{v\in E_1^0} v) = \sum_{v\in E_2^0} v$;*
2. *There is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism:*
$$K_0^{gr}(L_F(E_1)) \to K_0^{gr}(L_F(E_2)),$$
$$[L_F(E_1)] \mapsto [L_F(E_2)].$$
3. *There is a graded ring isomorphism $\psi : L_F(E_1) \to L_F(E_2)$.*

For a comprehensive treatment of Leavitt path algebras and the monoid M_E the reader is referred to the 2017 monograph [2], for many important results concerning the talented monoid T_E the reader is referred to the Hazrat and Li's 2020 work [5].

2.1 Statement of Authorship

The initial question of order unit equivalence was conceived by Prof. Roozbeh Hazrat. Anthony Warwick identified the equivalence criteria and worked with Prof. Hazrat on expressing these criteria in notation. Background work and proof for the confluence lemma has been summarised by Anthony Warwick from the literature though in some cases where the notation is quite succinct it has remained unchanged. All examples have been developed, solved and where applicable proved by Anthony Warwick.

3 Graph Monoids

Here we introduce some notation on directed graphs, the concepts of the monoids M_E, T_E and the monoid of the covering graph $M_{\overline{E}}$, along with the confluence lemma that shall be extensively used in solving examples and proofs.

3.1 Directed Graphs

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and two functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$ then the graph is called *row-finite*. A *path* μ in a graph E is a sequence of edges $\mu = e_1, e_2, ..., e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, ..., n-1$. In this case, $s(\mu) = s(e_1)$ is the *source* of μ, $r(\mu) = r(e_n)$ is the *range* of μ, and $n = |\mu|$ is the *length* of μ. If there is a path from a vertex u to a vertex v we write $u \geq v$. Let $\mu = e_1 e_2 ... e_n \in \text{Path}(E)$. If $n = |\mu| \geq 1$, and if $v = s(\mu) = r(\mu)$, then μ is a *closed path* based at v. A closed path $\mu = e_1 e_2 ... e_2$ such that $s(e_j) \neq v$ for every $j > 1$ is called a *closed simple path*. If $\mu = e_1 e_2 ... e_n$ is a closed path based at v and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle* based at v. We say that e is an *exit* for μ if there exists an $i (1 \leq i \leq n)$ such that $s(e) = s(e_i)$ and $e \neq e_i$. A graph is strongly connected if $u \geq v$ for all $u, v \in E^0$. We say the graph satisfies condition L if every cycle in E contains an exit. We say the graph satisfies condition K if and only if for each $v \in E^0$ which lies on a closed simple path there exist at least two distinct closed simple paths α, β based at v.

The covering graph $\overline{E}$ of E is defined by:

$$\overline{E}^0 = \{v_n \mid v \in E^0 \text{ and } n \in \mathbb{Z}\}, \qquad \overline{E}^1 = \{e_n \mid e \in E^1 \text{ and } n \in \mathbb{Z}\},$$
$$s(e_n) = s(e)_n, \qquad \text{and} \qquad r(e_n) = r(e)_{n+1}.$$

3.2 The Graph Monoid

The monoid M_E is defined as the free abelian monoid over the vertices in E^0 subject to identifying a vertex with the sum of vertices it is connected to:

$$M_E = \langle v \in E^0 \mid v = \textstyle\sum_{v \to u} u \rangle.$$

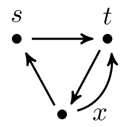
The talented monoid is defined as:

$$T_E = \langle v(i), v \in E^0, i \in \mathbb{Z} \mid v(i) = \textstyle\sum_{v \to u} u(i+1) \rangle.$$

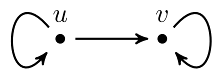
We define the algebraic pre-ordering on the monoid M by $a \leq b$ if $b = a + c$ for some $c \in M$. Notice the similarity to the definition of a path in the graph E.

Example 1. *M_E, T_E and the Monoid Covering Graph*

E:



F:



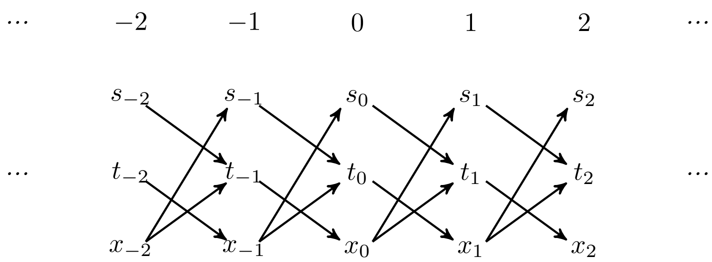
$$M_E = \frac{\langle s,t,x \rangle}{\left\langle \begin{matrix} s = t \\ t = x \\ x = s + t \end{matrix} \right\rangle}$$

$$M_F = \frac{\langle u,v \rangle}{\left\langle \; u = v \; \right\rangle}$$

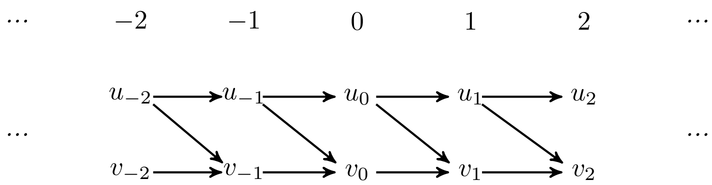
$$T_E = \frac{\langle s(i),t(i),x(i) \rangle}{\left\langle \begin{matrix} s(i) = t(i+1) \\ t(i) = x(i+1) \\ x(i) = s(i+1) + t(i+1) \end{matrix} \right\rangle}$$

$$T_F = \frac{\langle u(i),v(i) \rangle}{\left\langle \begin{matrix} u(i) = u(i+1) + v(i+1) \\ v(i) = v(i+1) \end{matrix} \right\rangle}$$

$\overline{E}$:



$\overline{F}$:



Notice that the covering graphs $\overline{E}, \overline{F}$ are acyclic, stationary graphs that "repeat" from "level" i to "level" $i+1$. We may refer to these levels as "shifts". $M_{\overline{E}}$ is used extensively in proof constructions as the monoid of the covering graph gives significant utility in keeping track of the number of vertices appearing within any given shift.

3.3 The Confluence Lemma

The following lemma [3] is crucial to the topic as it is how we shall establish all equivalences within our graph monoids.

Lemma 1. *Let E be a row-finite graph, F_E the free abelian monoid generated by E^0 and M_E the graph monoid of E.*

(1) *If* $a = a_1 + a_2$ *and* $a \to b$*, where* $a, a_1, a_2, b \in F_E \backslash \{0\}$*, then* b *can be written* $b = b_1 + b_2$ *with* $a_1 \to b_1$ *and* $a_2 \to b_2$.

(2) *(The Confluence Lemma) For* $a, b \in F_E \backslash \{0\}$*, we have* $a = b$ *in* M_E *if and only if there is* $c \in F_E \backslash \{0\}$ *such that* $a \to c$ *and* $b \to c$.

The talented version of the monoid M_E, denoted T_E, is then defined to be the abelian monoid generated by $\{v(i) \mid v \in E^0, i \in \mathbb{Z}\}$ for every v that emits edges and $i \in \mathbb{Z}$. Similarly to the M_E case (see **Appendix 1**), these relations define a congruence relation which also respects the action of $\mathbb{Z}$, namely T_E is equipped with a $\mathbb{Z}$ action

$$^{n}v(i) = v(i+n) \qquad i, n \in \mathbb{Z}.$$

Lemma 2. *Let* E *be a row-finite directed graph. We have the following morphisms between the monoids,* T_E*,* $M_{\overline{E}}$ *and* M_E*:*

1. *There is a forgetful homomorphism of monoids* ψ*:*

$$\psi\colon \quad T_E \to M_E,$$
$$v(i) \mapsto v.$$

2. *There is a* $\mathbb{Z}$*-module isomorphism of monoids* ϕ*:*

$$\phi\colon \quad T_E \to M_{\overline{E}},$$
$$v(i) \mapsto v_i.$$

Proof. 1 Since $\psi(u(i)) = u$ and $\psi(w(i)) = w \quad \forall u(i), w(i) \in T_E$
$\psi(u(i) + w(i)) = u + w = \psi(u(i)) + \psi(w(i))$
$\psi(e_{T_E}) = \psi(0) = 0 = e_{M_E}$
hence ψ is a monoid homomorphism. □

Proof. 2. Since $\phi(u(i)) = u_i$ and $\phi(w(i)) = w_i \quad \forall (u(i), w(i) \in T_E$
$\phi(u(i) + w(i)) = (u + w)_i = u_i + w_i = \phi(u(i)) + \phi(w(i))$
$\phi(e_{T_E}) = \phi(0) = 0 = e_{M_{\overline{E}}}$
hence ϕ is a monoid homomorphism. □

4 Leavitt Path Algebras

[2] Let E be an arbitrary directed graph and K any field. Define a set $(E^1)^*$ consisting of symbols of the form $\{e^* | e \in E^1\}$. The *Leavitt path algebra of* E *with coefficients in* K, $L_K(E)$, is the free associative K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the following relations:

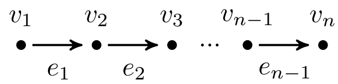
- (V) $\quad vv' = \delta_{v,v'}v \quad$ for all $v, v' \in E^0$,

- (E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
- (E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$,
- (CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e \in E^1$,
- (CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for every regular vertex $v \in E^0$.

4.1 Leavitt Path Algebras: Three Primary Colours

The following three examples are quoted from [2] as they are fundamental.

$A_n =$



oriented n-line graph

Let K be any field and $n \geq 1$ any positive integer. Then

$$M_n(K) \cong L_K(A_n)$$

Matrix Algebras.

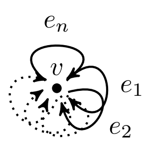
$R_1 =$

rose with one petal

$$K[x, x^{-1}] \cong L_K(R_1)$$

The *Laurent Polynomial K-algebra* i.e. the group algebra of $\mathbb{Z}$ over a field K. (Polynomials with integer coefficients).

$R_n =$



rose with n petals

Let K be any field and $n > 1$ any integer. The *Leavitt K-algebra of type* $(1, n), L_K(1, n)$ is the K-algebra:

$$K\langle X_1, ..., X_n, Y_1, ..., Y_n\rangle / \langle \sum_{i=1}^{n} X_iY_i - 1, Y_iX_j - \delta_{i,j}1 | 1 \leq i, j \leq n\rangle.$$

$$L_K(1, n) \cong L_K(R_n).$$

4.2 The $\mathbb{Z}$ Grading

There is a natural $\mathbb{Z}$ grading on Leavitt path algebras that is induced by the lengths of paths.

Definition 1. *Let G be a group and A an algebra over a field K. We say that A is G-graded if there exists a family $\{A_\sigma\}_{\sigma\in G}$ of K-subspaces of A such that*

$$A = \bigoplus_{\sigma\in G} A_\sigma \text{ as } K\text{-spaces, and } A_\sigma \cdot A_\tau \subseteq A_{\sigma\tau} \text{ for each } \sigma,\ \tau \in G.$$

An element x of A_σ is called a *homogeneous element of degree σ*. An ideal I of a G-graded K-algebra A is said to be a *graded ideal* if $I \subset \sum_{\sigma\in G}(I \cap A_\sigma)$, or equivalently if $y = \sum_{\sigma\in G} y_\sigma \in I$ implies $y_\sigma \in I$ for every $\sigma \in G$. In general, not every ideal in a Leavitt path algebra is graded.

4.3 Ideals

Let $H \subseteq E^0$. We say that H is *hereditary* if whenever $v \in H$ and $w \in E^0$, if $v \geq w$ then $w \in H$. We say that H is *saturated* if whenever $v \in Reg(E)$ has the property that $\{r(e)|e \in E^1, s(e) \in v\} \subseteq H$ then $v \in H$.
Every graded ideal I of $L_K(E)$ is generated by a hereditary and saturated subset of E^0, specifically, $I = I(I\cap E^0)$. There is a lattice isomorphism from the lattice of $\mathbb{Z}$-order ideals of T_E to the lattice of graded ideals $\mathcal{L}^{gr}(L_K(E))$ of $L_K(E)$:

$$\Phi : \mathcal{L}(T_E) \to \mathcal{L}^{gr}(L_K(E)$$

$$\langle H\rangle \mapsto I(H)$$

Where H is a hereditary and saturated subset of E^0, $\langle H\rangle$ is the order-ideal generated by the set $\{v \mid v \in H\}$ and $I(H)$ is the graded ideal generated by the same set.

4.4 The Graded non-IBN property in Leavitt Path Algebras

In [4] Hazrat identifies a criterion for the graded isomorphism of Leavitt Path Algebras:

Theorem 1. *There is a graded module isomorphism $L_n^k(\lambda_1, ..., \lambda_k) \cong_{gr} L_n^{k'}(\gamma_1, ..., \gamma_{k'})$ if and only if $\sum_{i=1}^{k} n^{\lambda_i} = \sum_{i=1}^{k'} n^{\gamma_i}$.*

it was shown in [5], **Remark 5.8** that there are $\mathbb{Z}$-module isomorphisms:

$$T_E \cong M_{\overline{E}} \cong \mathcal{V}(L_F(\overline{E})) \cong \mathcal{V}^{gr}(L_F(E)) \qquad 0.1$$

$$v(i) \mapsto v_i \mapsto [L_F(\overline{E})v_i] \mapsto [(L_F(E)v)(i)]$$

In 0.1 we now have that there is a $\mathbb{Z}$-graded isomorphism ϕ such that $\phi(1_E(\lambda_i)) = [L(E)\lambda_i]$.

Theorem 2. *Let E be a finite graph and T_E its associated $\mathbb{Z}$-graded graph monoid. Any finitely generated graded-projective $L(E)$-module is free if and only if for any $u \in E^0, u(i) = \sum_{i=1}^{k} 1_E(\lambda_i)$.*

Proof. The proof is immediate from correspondence (0.1).

□

5 Order Units of the Talented Monoid T_E

For a finite directed graph $E, \lambda \in \mathbb{Z}$, we denote by $1_E(\lambda)$ the element $\sum_{u \in E^0} u(\lambda) \in T_E$. We say that x is an *Order Unit* if for any $a \in T_E$ we have $a \leq \sum_{i=1}^{k} x(\lambda_i)$, where $\lambda_i \in \mathbb{Z}$. It is clear to see that 1_E is an order unit. Since we have $\phi(1_E(\lambda_i)) = [L(E)\lambda_i]$ from 0.1, the conditions of equivalence for order units yields significant information on the graded isomorphism critera for their corresponding Leavitt Path Algebras.

Definition 2. *Let E be a graph, $1_E \in T_E$ and let $\lambda, \gamma \in \mathbb{Z}$.*

1. *We say that 1_E is a* Fixed Order Unit *if* $\forall \ \sum_{i=1}^{k} 1_E(\lambda_i) = \sum_{i=1}^{k'} 1_E(\gamma_i) \implies k = k'$ *and* $\lambda = \gamma$.
2. *We say that 1_E is an* Alternating Order Unit *if* $\forall \ \sum_{i=1}^{k} 1_E(\lambda_i) = \sum_{i=1}^{k'} 1_E(\gamma_i) \implies k = k'$, *and* $\exists \ \sum_{i=1}^{k} 1_E(\lambda_i) = \sum_{i=1}^{k'} 1_E(\gamma_i)$ *such that* $\lambda \neq \gamma$.
3. *We say that 1_E is a* Replicating Order Unit *if* $\exists \ \sum_{i=1}^{k} 1_E(\lambda_i) = \sum_{i=1}^{k'} 1_E(\gamma_i)$ *such that* $k \neq k'$.

Theorem 1 *now reduces to study when 1_E is a Replicating Order Unit.*

It is important to note when solving examples that for any graph there are infinitely many solutions that partially fall under the Fixed Order Unit equivalence criteria i.e. $k = k'$ and $\lambda = \gamma$. This is due to the fact that for any sequence of transformations applied to the graph monoid T_E there is a sequence of "backwards" transformations that will simply reverse the applied transformations back to the initial conditions.

For each kind of equivalence we include examples from five different graph types identified from testing:

1. Strongly Connected Graphs
2. Graphs that are source free but not strongly connected:
   (a) Loop subgraphs.
   (b) Non-loop subgraphs.
3. Source containing graphs.

5.1 Fixed Order Unit

There do not appear to be any strongly connected graphs that are Fixed order units since strongly connected graphs occur in two configurations, either Condition L or not Condition L, and these configurations are Replicating order units and Alternating order units respectively.

The following is a demonstration of the beginnings of a rudimentary proof.

Theorem 3. *For a graph E and T_E its associated $\mathbb{Z}$-graded graph monoid, if 1_E contains a sink then T_E is a Fixed Order Unit.*

Proof. Suppose $v \in E^0$ is a sink, then $s^{-1}(v) = 0$ and there are no "forward" transformations possible such that $v \to_1 c$ for some $c \in E^0$. Suppose additionally that $r^{-1}(v) \neq 0$ then $v \in r(\alpha), \alpha \in Path(E)$, then there exists some $u \in \alpha$ such that when passing to T_E, $u(i) \geq v(j) \implies u(i) = v(j) + \sum_{k \in \mathbb{Z}} a(k)$. We have either:

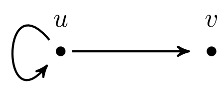
1. $a(k) = 0$ for all $u \in E^0, k \in \mathbb{Z}$ and there exist no bifurcated vertices in E^0 hence $u(i) \to v(j)$ for all $u \in E^0$ leaving the number of generators unchanged. Since there are no bifurcated vertices in E^0 we also have that there must exist some vertex u that is a source and $r^{-1}(u) = 0$. Passing now to $M_{\overline{E}}$, in order to consider a solution of the form $\sum_{i=1}^{k} 1_E(\lambda_i) = \sum_{i=1}^{k'} 1_E(\gamma_i)$ we must arrange to have every vertex in E^0 present at every shift within our canonical presentation. Since we cannot modify the total number of generators by any transformations, there exists a source vertex u where the only possible transformations strictly increase the subscript and a sink vertex v where the only possible transformations strictly decrease the subscript the only possible presentation of $\sum_{i=1}^{k} 1_E(\lambda_i) = \sum_{i=1}^{k'} 1_E(\gamma_i)$ occurs when $k = k'$ and $\lambda = \gamma$ where all transformations from LHS "undo" any transformations from RHS.
2. $a(k) \neq 0$ for some $u \in E^0, k \in \mathbb{Z}$. There exists some bifurcated vertex $u \in \alpha$. Incomplete.

□

The case containing bifurcated vertices is more complicated than the case with no bifurcated vertices. This reflects the complexity of working with the Confluence lemma where we must be able to pass elements both "backward" and "forward" while simultaneously keeping track of the shifts and the number of generators.

Example 2. *Fixed Order Unit*

E:

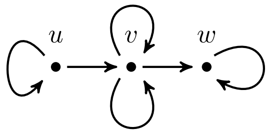


$$T_E = \frac{\left\langle u(i), v(i) \right\rangle}{\left\langle \;\; u(i) = u(i+1) + v(i+1) \;\; \right\rangle}$$

$$\underbrace{1_E(0) + 1_E(1)}_{k=2,\lambda_1=0,\lambda_2=1} \to^{1_E(0)} u(1) + v(1) + v(0) + u(1) + v(1) \leftarrow^{u(0)} u(1) + v(1) + v + u = \underbrace{1_E(0) + 1_E(1)}_{k'=2,\gamma_1=0,\gamma_2=1}$$

Example 3. *Fixed Order Unit: No Sink*

E:



$$T_E = \frac{\langle u(i), v(i, w(i)) \rangle}{\left\langle \begin{array}{l} u(i) = u(i+1) + v(i+1) \\ v(i) = 2 \cdot v(i+1) + w(i+1) \\ w(i) = w(i+1) \end{array} \right\rangle}$$

$$\underbrace{1_E(0) + 3 \cdot 1_E(1)}_{k=4, \lambda_1=0, \lambda_2=1} \rightarrow^{1_E(0)} u(1) + 3 \cdot v(1) + 2 \cdot w(1) + 3 \cdot 1_E(1)$$
$$\leftarrow^{u(0)} u(1) + 3 \cdot v(1) + 2 \cdot w(1) + u(0) + 2 \cdot u(1) + 2 \cdot v(1) + 3 \cdot w(1)$$
$$\leftarrow^{v(0)} u(1) + 3 \cdot v(1) + 2 \cdot w(1) + u(0) + 2 \cdot u(1) + v(0) + 2 \cdot w(1)$$
$$\leftarrow^{w(0)} 3 \cdot u(1) + 3 \cdot v(1) + 3 \cdot w(1) + u(0) + v(0) + w(0) = \underbrace{3 \cdot 1_E(1) + 1_E(0)}_{k'=4, \gamma_1=0, \gamma_2=1}$$

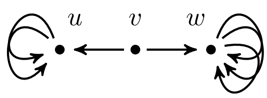
While this example may seem trivial as we simply "undo" any transformations, this is the form that all fixed order unit solutions take. Notice that the only possible transformations we can apply (either "forward" or "backward") to u leave the total number of u generators the same but change the shift level, while there are transformations that increase the number of generators v and w. After the initial transformation step we still have 4 u generators, while we have 6 v generators and 5 w generators. The form of the partial solution with an "incomplete" 1_E then requires us to find these additional generators u (and w) such that we can achieve a solution that can be completely expressed in terms of 1_E.

While the number of generators u remains the same, any transformation of u will alter the number of v generators effectively "fixing" the possible shift level those u generators are capable of appearing at in the covering graph monoid $M_{\overline{E}}$.

If it were the case that we were only required to find additional generators w we would not have this problem since there are transformations $w(i) \rightarrow_1 w(i+1) \rightarrow_k w(i+k)$ so the shift of w could be arranged to appear on any level in $M_{\overline{E}}$, we will see examples of graphs of this form when looking at alternating order units.

Example 4. *Fixed order unit: Source Containing*

F:



$$T_E = \frac{\langle u(i), v(i), w(i) \rangle}{\left\langle \begin{array}{l} u(i) = 2 \cdot u(i+1) \\ v(i) = u(i+1) + w(i+1) \\ w(i) = 3 \cdot w(i+1) \end{array} \right\rangle}$$

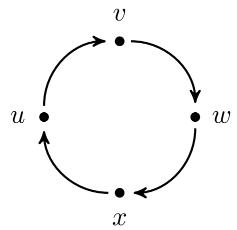
$$\underbrace{1_F(0) + 4 \cdot 1_F(1)}_{k=5, \lambda_1=0, \lambda_2=1} = \underbrace{4 \cdot 1_F(1) + 1_F(0)}_{k'=5, \gamma_1=1, \gamma_2=0}$$

For neatness the talented monoids for the remainder of the examples will be omitted as they are easy to derive.

5.2 Alternating Order Unit

Example 5. *Alternating Order Unit: Strongly Connected*

E:



$$\underbrace{1_E(i)}_{k=1,\lambda_1=i} \rightarrow^{j\cdot 1_E(i)} u(i+j)+v(i+j)+w(i+j)+x(i+j) = \underbrace{1_E(i+j)}_{k'=1,\gamma_1=i+j}$$

Theorem 4. *For a graph E and T_E its associated $\mathbb{Z}$-graded graph monoid, if E is Strongly Connected and there exists a cycle with no exit then 1_E is an Alternating Order Unit.*

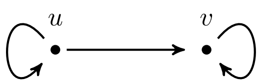
Proof. Suppose E is strongly connected so $\mathcal{H} = \{\emptyset, E^0\}$ and $u \geq v$ for any $u, v \in E^0$. In this case E is also cofinal. Thus if there exist cycles in E^0 then every vertex in E^0 connects to a cycle. Additionally since the only order ideals in E^0 are trivial it must be the case that every vertex $v \in E^0$ is contained within the cycle c^0 else there exists some hereditary and saturated proper subset $H \subset E^0$ that is not empty.

Now since all $v \in E^0$ are contained within a cycle with no exit it is the case that we have ${}^n v = v$ for all $v \in E^0$ and there is only one edge emitting from each vertex. Thus any transformations we apply to $\sum_{i=1}^{k} 1_E(\lambda_i)$ will leave the number of generators unchanged but we can apply transformations to arrange for every vertex in $1_E(\lambda_i)$ to appear at any shift, hence $k = k'$ and there exists a solution where $\lambda \neq \gamma$.

□

Example 6. *Alternating Order Unit: Source Free a)*

F:

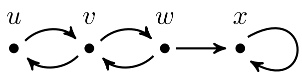


$$\underbrace{1_F(0)+1_F(2)}_{k=2,\lambda_1=0,\lambda_2=2} \rightarrow^{1_E(0)} u(1)+v(1)+v(1)+u(2)+v(2) \leftarrow^{u(1)} u(1)+v(1)+u(1)+v(1) = \underbrace{2\cdot 1_E(1)}_{k'=2,\gamma_1=1}$$

For Source Free a) cases it appears the differentiating factor from other equivalence types is the existence of a vertex $v \in E^0$ that is not in the range of a vertex that can generate additional copies of itself while also being "sufficiently close" to another vertex such that $\mathbb{Z}$ acts freely on that vertex, subsequently allowing backward transformations that can be from any "age".

Example 7. *Alternating Order Unit: Source Free b)*

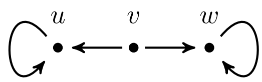
E:



$$\underbrace{1_E(0)+1_E(1)+2\cdot 1_E(i)+1_E(i-1)}_{k=5,\lambda=\{0,1,i,i-1\}} = \underbrace{1_E(1)+2\cdot 1_E(2)+1_E(i-2)+1_E(i-1)}_{k'=5,\gamma=\{1,2,i-2,i-1\}}$$

Example 8. *Alternating order unit: Source Containing*

F:

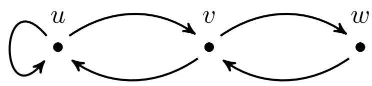


$$\underbrace{1_F(0)}_{k=1,\lambda=0} = \underbrace{1_F(1)}_{k'=1,\gamma=1}$$

5.3 Replicating Order Unit

Example 9. *Strongly connected graph*

E:



$$\underbrace{1_E(0)+1_E(3)}_{k=2,\lambda_1=0,\lambda_2=3} \rightarrow^{1_E(0)} 2\cdot u(1)+2\cdot v(1)+w(1)+1_E(3)$$

$$\rightarrow^{u(1)}_{v(1)} u(1)+v(1)+w(1)+2\cdot u(2)+v(2)+w(2)+1_E(3)$$

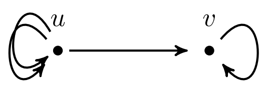
$$\leftarrow^{v(2)}_{w(2)} u(1)+v(1)+w(1)+2\cdot u(2)+2\cdot v(2)+2\cdot w(2) = \underbrace{1_E(1)+2*1_E(2)}_{k'=3,\gamma_1=1,\gamma 2=2}$$

Conjecture 2. *For a strongly connected graph E, if E satisfies condition L then 1_E is a replicating order unit.*

Note that in this case we have $u \geq v \quad \forall \quad u,v \in E^0$ which becomes, in T_E, $u(i) = v(i+j) + \sum_{k\in\mathbb{Z}} \beta(k)$, $\beta(k) \neq 0$ for some $k \in \mathbb{Z}$. We can see that when we apply any transformation to any vertex within a graph of this type that we will always increase the number of generators suggesting that we have a solution of type $k \neq k'$. Similarly to the case of fixed order units with bifurcations the proof is non-trivial due to needing to work both "backwards" and "forwards" with the confluence lemma as we can see in **Example 9.**.

Example 10. *Source Free Graph a)*

F:



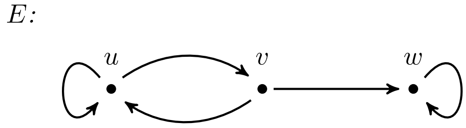
$$1_F(0) = u(0)+v(0) \rightarrow^{u(0)} 2u(1)+v(1)+v(0)$$

$$\rightarrow^{v(0)} 2\cdot u(1)+2\cdot v(1) = 2\cdot 1_F(1)$$

Theorem 5. *If $|r^{-1}(v)| = n$ for all $v \in E^0$ then $1_E \in T_E$ is a Replicating Order Unit.*

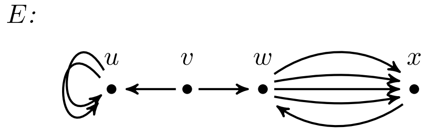
Proof. Suppose $|r^{-1}(v)| = n$ for all $v \in E^0$. Apply a single $\to_1$ transformation to every $v \in \sum_{i=1}^{k} 1_E(\lambda_i)$. Since the number of any given generator $v(i)$ in $M_{\overline{E}}$ increases by the $|r^{-1}(v(i))|$ and this equals n for all $v \in E^0$ then we have that $\sum_{i=1}^{k} 1_E(\lambda_i) = n \cdot \sum_{i=1}^{k} 1_E(\lambda_i + 1) = \sum_{i=1}^{k'} 1_E(\gamma_i)$. □

Example 11. *Source Free Graph b)*



$$\underbrace{1_E(0) + 1_E(1) + 1_E(2)}_{k=3} = \underbrace{2 \cdot 1_E(1) + 2 \cdot 1_E(2)}_{k'=4}$$

Example 12. *Source containing graph*

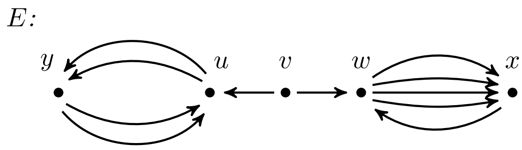


$$\underbrace{1_E(0)}_{k=1} = \underbrace{2 \cdot 1_E(1)}_{k'=2}$$

Nam and Phuc [6] give a criterion for the (non-graded) IBN property for rings stating that after a process of source eliminations if the source free graph E_{sf} contains any isolated vertexes or source cycles then the graph E has IBN property. To investigate the non-graded IBN property it appears that we may apply similar method. As opposed to conducting repeated source eliminations and investigating structures within the isolated sub-graphs we look for order unit equivalences within minimal ideals of the talented monoid, requiring each minimal ideal to be a replicating order unit, though a specific proof has yet to be identified.

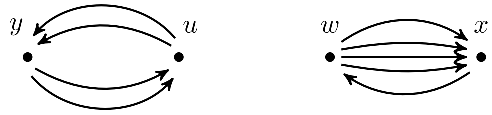
From examples studied it appears that there are conditions on the sub-ideals within E. After conducting source eliminations we attempt to find a correspondence between the order units of each minimal ideal, where the sub-graph of each minimal ideal must be a replicating order unit. I shall demonstrate this specifically in the following examples.

Example 13. *Source-Free*



$$1_E(0) \to 4 \cdot 1_E(2)$$

E^{min}_{sf}:



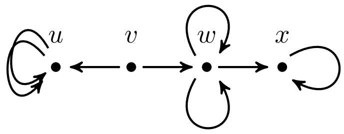
Let $I = \langle u, y \rangle$ *Let* $J = \langle w, x \rangle$

$$1_I(0) \to 2 \cdot 1_I(1) \to 4 \cdot 1_I(2)$$

$$1_J(0) \quad \to \quad 4 \cdot 1_J(2)$$

Example 14. *Non-replicating Source Containing Graph*

E:



$$\underbrace{1_E(0) + 3 \cdot 1_E(1)}_{k=4, \lambda_1=0, \lambda_2=1} = \underbrace{1_E(0) + 3 \cdot 1_E(1)}_{k'=4, \gamma_1=0, \gamma_2=1}$$

Working: Start with $1_E(0)$ alone to identify trends.

$$1_E(0) \to^{1_E(0)} 3 \cdot u(1) + 3 \cdot w(1) + 2 \cdot x(\alpha) \to^{u(1)}_{w(1)} 2 \cdot u(1) + 2 \cdot w(1) + 3 \cdot x(\alpha) + 2 \cdot u(2) + 2 \cdot w(2)$$

$$= 2 \cdot 1_E(1) + x(i)$$

Notice the shift action on $x(\alpha)$ can take any value.

$$u(i+1) + w(i+1) \leftarrow v(i)$$

We appear to be generating additional $x(\alpha)$ over the other generators.

Having seen examples such as this before, these kinds of solution appear to require "backwards" transformations from other copies of $1_E(i)$ to "fill in the gaps" i.e. $1_E(0) \to c$ and $d \leftarrow \sum_{i=1}^{k} 1_E(\lambda_i)$ such that $c + d = \sum_{j=1}^{k'} 1_E(\gamma_j)$.

$$3 \cdot 1_E(i) \leftarrow^{w(i-1)} 3 \cdot u(i) + 3 \cdot v(i) + w(i) + w(i-1) + 2 \cdot x(i) \leftarrow^{u(i-1)}$$
$$u(i) + u(i-1) + 3 \cdot v(i) + w(i) + w(i-1) + 2 \cdot x(i) \leftarrow^{v(i-1)}$$
$$u(i-1) + v(i-1) + w(i-1) + 3 \cdot v(i) + 2 \cdot x(i) \leftarrow^{x(i-1)}$$
$$1_E(i-1) + 3 \cdot v(i) + x(i)$$

From above:

$$1_E(0) \to^{1_E(0)} 3 \cdot u(1) + 3 \cdot w(1) + 2 \cdot x(\alpha)$$

Setting the initial shift of our "fill in the gaps" transformations to 1 and substituting into the equation:

$$1_E(0) + 3 \cdot 1_E(1) \to^{1_E(0)} 3 \cdot u(1) + 3 \cdot w(1) + 2 \cdot x(1) + 3 \cdot 1_E(1) \leftarrow$$
$$3 \cdot u(1) + 3 \cdot w(1) + 2 \cdot x(1) + 1_E(0) + 3 \cdot v(1) + x(1) = 1_E(0) + 3 \cdot 1_E(1)$$

The question then becomes whether *all* solutions for this particular graph appear in this form where $k = k'$ and $\lambda = \gamma$.

Breaking the graph E into minimal order ideals we have:

E_{sf}^{min}:

Let $I = \langle u \rangle$ *Let* $J = \langle x \rangle$

$$\underbrace{1_I(0)}_{k=1,\lambda_1=0} \rightarrow \underbrace{2 \cdot 1_I(1)}_{k'=2,\gamma_1=1} \qquad \textit{Replicating.}$$

$$\underbrace{1_J(0)}_{k=1,\lambda_1=0} \rightarrow 1_J(1) \rightarrow \underbrace{1_J(i)}_{k'=1,\gamma_1=i} \qquad \textit{Alternating.}$$

While this solution may be an indication that the order unit equivalence type of this graph is fixed, recall that all graphs have infinitely many solutions of this form.

This highlights some of the problems with attempting to identify graph criteria for these equivalences. It is possible to find "apparently fixed" solutions for any graph. In many cases I have found that examples I had previously classified as Fixed or Alternating order units were in fact Replicating under closer inspection. In most of these cases I found this while attempting to provide an if and only if proof for an identified criteria.

6 Discussion and Conclusion

We identified a simple proof on the nature of free finitely generated graded-projective modules that relates the structure of the talented monoid of a finite graph to its Leavitt path algebra through an order unit condition. Additionally, we identify several order unit equivalence criteria suggesting that the talented monoid and its associated grading capture significant structural information of the corresponding Leavitt path algebra. Both of these results suggest that there is further work in this direction that may contribute to our understanding of the Graded Hirchsberg-Phillips conjecture. Due to the complexity of working with these equivalence criteria it would be advantageous to determine the graph criteria for fixed order unit equivalence first as this appears to be the simplest case. We could then expand upon that work into determining the algebraic correspondence for the fixed order unit condition which is unknown, and continue work on the more complicated equivalence criteria.

Leavitt path algebras are an important tool for considering not necessarily linear systems while also having a relatively simple presentation via the talented monoid, showing promise that further developments may provide valuable insight into other domains of mathematics, computer science and physics.

7 Acknowledgements

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8 References

References

[1] Abrams, G & AP, G 2005, 'The Leavitt path algebra of a graph', *Journal of algebra*, vol. 293(2), pp. 319-334.

[2] Abrams, G, Ara, P & Molina MS 2017, 'Leavitt path algebras', *Lecture notes in mathematics*, vol. 2191, Springer-Verlag, London, UK.

[3] Ara, P, Moreno, MA & Pardo, E 2007, 'Nonstable K-theory for graph algebras', *Algebras and representation theory*, vol. 10, pp. 157-178.

[4] Hazrat, R 2013, 'The graded Grothendieck group and the classification of Leavitt path algebras', *Mathematische annalen*, 355(1), pp. 273-325.

[5] Hazrat, R & Li, H 2020, 'The talented monoid of a Leavitt path algebra', *Journal of algebra*, vol. 547, pp. 430-455.

[6] Nam, TG & Phuc, NT 2019, 'The structure of Leavitt path algebras and the invariant basis number property', *Journal of pure and applied algebra*, vol. 223(11), pp. 4827-4856.

8.1 Bibliography

Ara, P, Pardo, E 2014, 'Towards a K-theoretic characterization of graded isomorphisms between Leavitt path algebras', *Journal of K-theory*, vol. 14(2), pp. 203-245.

Hazrat, R 2016, 'Graded rings and graded Grothendieck groups', *London mathematical lecture notes series*, vol. 435, Cambridge University Press, Cambridge.

9 Appendices

9.1 The Confluence Lemma: Proof

Let M_E be the free abelian monoid on the set E^0. The nonzero elements of M can be written uniquely up to permutation as $\sum_{i=1}^{n} v_i$, where $v_i \in E^0$. For convenience we introduce the notation: For $v \in E^0$, write

$$\mathbf{r}(v) := \sum_{e \in E^1 \ | s(e)=v} r(e) \in M$$

Now our relations in M_E becomes $v = \mathbf{r}(v)$ for every $v \in E^0$ that emits edges.

We define a binary relation $\to_1$ on $M\backslash\{0\}$, $\sum_{i=1}^n v_i \in M$, $j \in \{1, ..., n\}$ such that v_j emits edges (i.e. all vertices $i \neq j$ do not emit any edges) by $\sum_{i=1}^n v_i \to_1 \sum_{i\neq j} v_i + \mathbf{r}(v_j)$. Let $\to$ be the transitive and reflexive closure of $\to_1$ on $M\backslash\{0\}$ i.e. $\alpha \to \beta$ if and only if there is a finite string $\alpha = \alpha_0 \to_1 \alpha_1 \to_1 ... \to_1 \alpha_n = \beta$. Let $\sim$ be the congruence on M generated by the relation $\to_1$ or $\to$. Specifically, $\alpha \sim \alpha$ for all $\alpha \in M$ and for $\alpha, \beta \neq 0$ we have $\alpha = \beta$ iff there is a finite string of transformations $\alpha = \alpha_0, \alpha_1, ..., \alpha_n = \beta$ such that for each $i = 0, 1, ..., n-1$ either $\alpha_i \to_1 \alpha_{i+1}$ or $\alpha_{i+1} \to_1 \alpha_i$. We call n the *length* of the string of transformations. Since $\sim$ is the congruence on M generated by the relations on M_E we have $M_E = M\backslash \sim$. The *support* of an element γ in M $supp(\gamma) \subseteq E^0$ is the set of basis elements appearing in the canonical expression γ.

Lemma 3. *Let E be a row-finite graph, F_E the free abelian monoid generated by E^0 and M_E the graph monoid of E.*

(1) *If $a = a_1 + a_2$ and $a \to b$, where $a, a_1, a_2, b \in F_E\backslash\{0\}$, then b can be written $b = b_1 + b_2$ with $a_1 \to b_1$ and $a_2 \to b_2$.*

(2) *(The Confluence Lemma) For $a, b \in F_E\backslash\{0\}$, we have $a = b$ in M_E if and only if there is $c \in F_E\backslash\{0\}$ such that $a \to c$ and $b \to c$.*

Proof. (1) By induction it is sufficient to show the result for the case $a \to_1 b$. There is an element u in the support of a such that we may write $a = (a - u) + \mathbf{r}(u)$. Now if $a \to_1 b$ we may write $b = (a - u) + \mathbf{r}(u)$. Consider the case where u belongs either to the support of a_1 or to the support of a_2. Assume for instance that u belongs to the support of a_1. Then we simply set $b_1 = (a_1 - u) + \mathbf{r}(u)$ and $b_2 = a_2$. □

Note that the elements b_1 and b_2 are not uniquely determined by a_1 and a_2 in general. This is due to the possibility of u belonging to the support of both a_1 AND a_2.

Proof. (2) Assume $a \sim b$. So there exists a finite string $a = a_0, a_1, ..., a_n = b$ such that for each $i = 0, 1, 2, ..., n-1$ either $a_i \to_1 a_{i+1}$ or $a_{i+1} \to_1 a_i$. By induction on n, if $n = 0$ then $a = b$ and there is nothing to prove. Assume the result is true for strings of length $n-1$ and let $a = a_0, a_1, ..., a_n = b$ be a string of transformations of length n. By our induction supposition there is a $d \in M$ such that $a \to d$ and $a_{n-1} \to d$ and there are now two cases to consider. If $b \to_1 a_{n-1} \to d$ and we are done. Assume that $a_{n-1} \to_1 b$. There is an element x in the support of a_{n-1} and we can arrange such that $a_{n-1} = x + a'_{n-1}$ where all of the vertices in E^0 that emit edges appear in the canonical expression of the element x. Now by the definition of $\to_1$ we have $b = \mathbf{r}(x) + a'_{n-1}$. By part (1) of the lemma we can write $d = d(x) + d'$ where $x \to d(x)$ and $a'_{n-1} \to d'$. Now if the length of the string of transformations from x to $d(x)$ is positive we have $\mathbf{r}(x) \to d(x)$ and so $b = \mathbf{r}(x) + a'_{n-1} \to d(x) + d' = d$ and we are done. In the case where the length of the string is zero i.e. $x = d(x)$ we can select an element c such that we have $d = x + d' \to_1 \mathbf{r}(x) + d' = c$ and so we have from earlier $a \to d \to_1 c$ and also $b = \mathbf{r}(x) + a'_{n-1} \to \mathbf{r}(x) + d' = c$ concluding the proof. □