AMSI VACATION RESEARCH SCHOLARSHIPS 2021–22

Get a taste for Research this Summer

The Music of the Spheres: Spectral Sequences

Olin Gao

Supervised by Vigleik Angeltveit

Australian National University

Vacation Research Scholarships are funded jointly by the Department of Education and the Australian Mathematical Sciences Institute.

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Abstract

Our goal is to exploit the computational power of the Serre and Adams spectral sequences to understand the stable homotopy groups of spheres. We will begin with a general derivation of spectral sequences. Then we will proceed to explain patterns which arise in the homotopy groups of spheres using the Serre spectral sequence. We will conclude by discussing how to construct the Adams spectral sequence using the Steenrod algebra and comment on the information it gives us about maps between spheres of different dimensions.

1 Philosophy and Introduction

A spectral sequence is essentially an algorithm that we can associate with a topological object, and there exists a state to which this algorithm converges. Spectral sequences are in many cases the best tool to gather insights about these topological objects. For example, it was only through the study of the Adams spectral sequence that the famous result about the non-existence of 16 dimensional division algebras was proven by examining patterns in the maps between spheres. This report will focus on understanding these remarkable patterns which arise in the world of spheres. Historically, topologists have made the analogy that the diagram of these patterns is like 'the musical score of the spheres'. *This report assumes knowledge up to Chapter 3: Cohomology in Allen Hatcher's "Algebraic Topology" and some basic definitions in homotopy theory.*

2 Statement of Authorship

The results and ideas in this report primarily come from Allen Hatcher's unreleased textbook on spectral sequences, John McCleary's "A User's Guide to Spectral Sequences", and of course, Vigleik Angeltveit.

All maps in this report are continuous unless otherwise specified. All chain groups are finitely generated. Objects with undefined indices are 0. $\circ$ *is omitted in function composition.*

3 Preliminary Topological Ideas

X is an n-connected space if it is path connected and its first n homotopy groups vanish. That is, $\pi_i(X) \cong 0$ for $1 \leq i \leq n$.

Theorem 3.1. *Hurewicz Theorem. If X is $n-1$-connected, then there exists an isomorphism $\rho_n : \pi_n(X)_{ab} \to H_n(X)$, recalling that $\pi_n(X)_{ab} \cong \pi_n(X)$ for $n \geq 2$.*

Recall that for $n > 0$, we have that $H_n(X) \cong H_{n+1}(\Sigma X)$ where $\Sigma X \cong \frac{X \times [0,1]}{(x,0)\sim(y,0),\,(x,1)\sim(y,1)}$. In an *adjoint* sense, $\pi_{n+1}(X) \cong \pi_n(\Omega X)$ where $\Omega X = \{\gamma : S^1 \to X\}$.

Theorem 3.2. *Existence of Eilenberg-Maclane spaces. For any group G and $n \in \mathbb{Z}_+$, there exists a space $K(G,n)$ with the property that* $\pi_i(K(G,n)) \cong \begin{cases} G \text{ if } i = n \\ 0 \text{ otherwise} \end{cases}$.

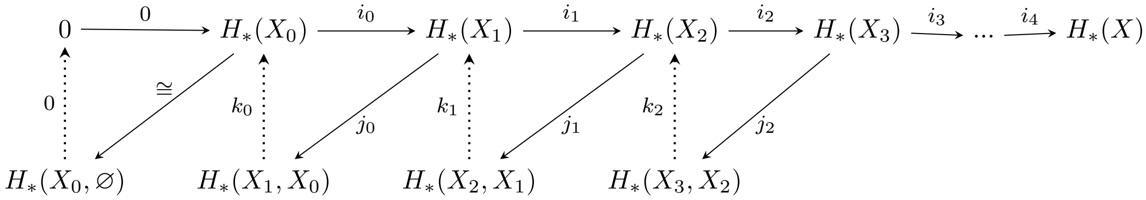
4 Deriving the Spectral Sequence

We will cover the core ideas which allow spectral sequences to exist in nature.

Suppose we have a filtration of X (whose union is X):

$$\varnothing \subseteq X_0 \subseteq X_1 \subseteq X_1 \subseteq ... \subseteq X$$

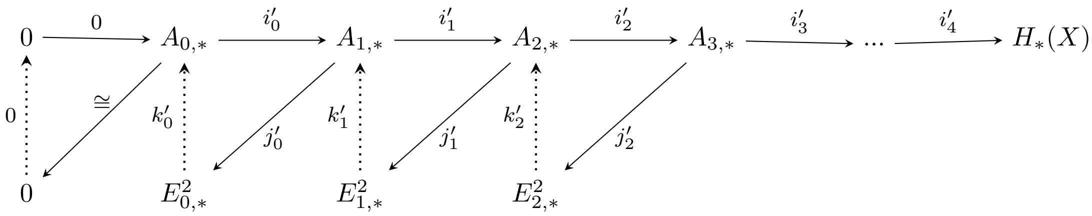
Then the long exact sequence of a pair gives us the following diagram.



Such a diagram is called an **exact couple**. Note that each cycle formed by the maps $k_r j_r i_r$ represents a long exact sequence. That is, maps labelled i and j indicate maps between homology groups of the same dimension, and maps labelled k reduce homology dimension by 1.

Let $E^1_{s,t} := H_{s+t}(X_s, X_{s-1})$ and define $d^1_{s,t} : E^1_{s,t} \to E^1_{s-1,t}$ by $d^1_{s,t} = j_{s-2}k_{s-1}$. We have that $d^1_{s-1,t}d^1_{s,t} = 0$ by exactness so that the $(E^1_{s,t}, d^1_{s,t})$ form a chain complex in s. Hence we can take homology, defining $E^2_{s,t} = \frac{\ker(d^1_{s,t})}{\text{im}(d^1_{s+1,t})}$. Now we arrive at the heart of spectral sequences:

Theorem 4.1. *Let the A_s be the images of the inclusion maps i_s. Then the following diagram has the same exactness properties as the exact couple. Note that we define $j'(ia) = [ja]$, hence the offset in the direction of the maps.*



Proof. We will take for granted that the i', j' and k' are well defined. The 6 conditions to prove for exactness are $j'i' = 0$, $\ker j' \subset \text{im } i'$, $k'j' = 0$, $\ker k' \subset \text{im } j'$, $i'k' = 0$ and $\ker i' \subset \text{im } k'$. Of these, $\ker j' \subset \text{im } i'$ is the most involved so we will just demonstrate that. We have, in condensed form, $j'a' = 0, a' = ia \Rightarrow [ja] = j'a' = 0 \Rightarrow ja \in \text{im } d \Rightarrow ja = jke \Rightarrow a - ke \in \ker j = \text{im } i \Rightarrow a - ke = ib \Rightarrow i(a - ke) = ia = i^2 b \Rightarrow a' = ia \in \text{im } i_2 = \text{im } i'$. □

In essence, this result means that we can keep repeating the process of taking homology by considering the $E^r_{s,t}$ as chain complexes with differentials $d^r = jk$. This sequence of lattices E^r (indexed by integers s, t) and differentials d^r is known as a **spectral sequence**. By convention, we call the (E^r, d^r) step the r^{th} **page**.

5 Convergence

Noting that the map labelled j was defined to map between different lattice points on each page, we deduce that the differentials $d^r = jk$ must map from $E^r_{s,t}$ to $E^r_{s-r,t-r+1}$.

Theorem 5.1. *There exists a page E^∞ where for any integers s, t, there exists $R \in \mathbb{Z}_+$ such that $E^r_{s,t} = E^\infty_{s,t}$ for all $r \geq R$.*

Proof. For any s, t, we have that the differentials to and from $E^r_{s,t}$ will eventually map outside the quadrant where $s, t \geq 0$ where all groups are 0 by definition. □

Theorem 5.2. *Let (E^r, d^r) be the spectral sequence for a filtration $\varnothing \subseteq X_0 \subseteq X_1 \subseteq \ldots$ of X. Define $F^t_s = \mathrm{im}\,(\ldots \circ i_{t+1} \circ i_t : H_s(X_t) \to H_s(X))$. If only finitely many terms in each column $E^1_{s,*}$ are non-zero, then $E^\infty_{s,t}$ is isomorphic to F^t_s / F^{t-1}_s for the filtration $0 \subseteq F^0_s \subseteq F^1_s \subseteq \ldots$ of $H_s(X)$.*

The upshot is that the E^∞ page reveals information about the homology of our original space X. A special case is that if X is contractible, then the E^∞ page is blank except at $E^\infty_{0,0}$.

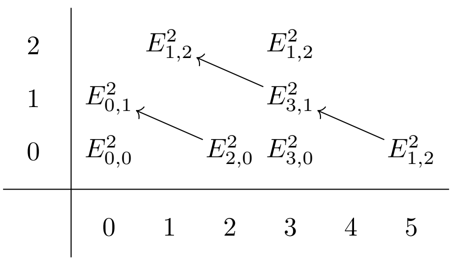
6 The Serre Spectral Sequence

Theorem 6.1. *If $F \to X \to B$ is a fibration, then for sufficiently nice F we have that there exists a spectral sequence (E^r, d_r) with*

1. *$E^2_{p,q} \cong H_p(B; H_q(F; G))$.*
2. *$E^\infty_{p,n-p}$ is isomorphic to the successive quotients F^p_n / F^{p-1}_n in a filtration $0 \subseteq F^0_n \subseteq \ldots \subseteq F^n_n = H_n(X; G)$ of $H_n(X; G)$.*

Proof. Consider the long exact sequence on fibrations. □

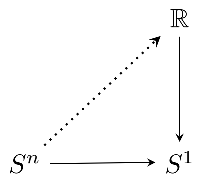
First, note that our notation conventions change slightly in different contexts. The history of spectral sequences is very rich and mathematicians from all different walks of life have their own notational preferences. Below is how the E^2 page is typically visualised (with the arrows representing the d_2).



For intuition's sake, we can think of $H_p(B; H_q(F; G)) \cong H_p(B; \mathbb{Z})$"$\otimes$"$H_q(F; G)$. If the coefficient G is a field then the tensor product always holds, but in general instead of just the tensor you have the direct sum with the derived functor of the tensor product. Further, for the purposes of this report (and our sanity) we can simply think of $H_n(X; G)$ as being isomorphic to $\bigoplus_p E^\infty_{p,n-p}$.

6.1 Sample Calculation

Let's do a simple calculation with this new fangled computational tool. Consider the fibration $S^1 \to P \xrightarrow{f} K(\mathbb{Z}, 2)$ where P is the space of paths in $K(\mathbb{Z}, 2)$. To see how this is a fibration, convince yourself that the fibre of f is $\Omega K(\mathbb{Z}, 2)$, then by properties of the loop space, we have $K(\mathbb{Z}, 1) \cong \Omega K(\mathbb{Z}, 2)$, then S^1 is a $K(\mathbb{Z}, 1)$ since $\pi_n(S^1) \cong 0$ for $n > 1$ by the lifting criterion:



Now, letting $B := K(\mathbb{Z}, 2)$ and taking $\mathbb{Z}$ to be our homology coefficient, we have the E^2 page of the Serre spectral sequence:

1	$\mathbb{Z}$	$H_1(B)$	$H_2(B)$	$H_3(B)$	$H_4(B)$	$H_5(B)$	$\cdots$
0	$\mathbb{Z}$	$H_1(B)$	$H_2(B)$	$H_3(B)$	$H_4(B)$	$H_5(B)$	$\cdots$
	0	1	2	3	4	5	6

Now, since P is contractible, the E^∞ page must have 0 in every coordinate except $(0, 0)$. But since $d_n = 0$ for $n > 2$ due to the lack of non-zero groups above the row $t = 1$, we have that $E^3 \cong E^\infty$. This means that all sequences of groups connected by differentials in E^2 must form exact sequences except at $(0, 0)$. Using the exact sequences $0 \to H_1(B) \to 0$ and $0 \to H_2(B) \to \mathbb{Z} \to 0$ then induction, we deduce that

$$H_n(B) \cong H_n(K(\mathbb{Z}, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

These look like the homology groups of $\mathbb{C}P^\infty$. Indeed, $\mathbb{C}P^\infty$ satisfies the properties to be a $K(\mathbb{Z}, 2)$.

7 The Harmony of the Spheres

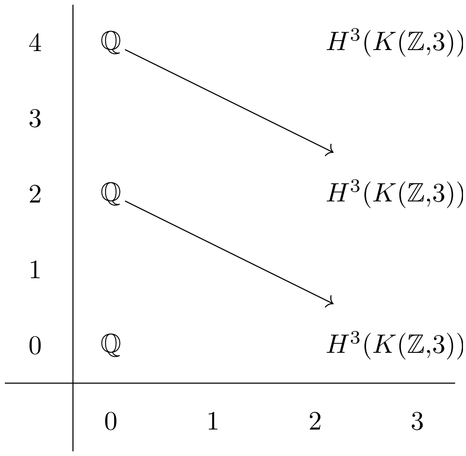
This section will be dedicated to justifying the following pattern found in the homotopy groups of spheres.

Theorem 7.1.

$$\pi_k S^n = \begin{cases} 0 \textit{ if } k < n \\ \mathbb{Z} \textit{ if } k = n \\ \mathbb{Z} \oplus G_{finite} \textit{ if } n \textit{ even, } k = 2n - 1 \\ G_{finite} \textit{ otherwise} \end{cases}$$

The first two cases are immediate from elementary results in algebraic topology. At this point we should probably mention that although we have built our theory of spectral sequences on the shoulders of homology, moving to cohomology is a completely seemless process and all our previous calculations and justifications still hold (by appropriately reversing directions of maps).

First, we propose that $H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[x_n] \text{ if } n \text{ even} \\ \Lambda_{\mathbb{Q}}(x_n) \text{ if } n \text{ odd} \end{cases}$, recalling that $\Lambda_{\mathbb{Q}}(x_n)$ is the alternating algebra (so $x_n^2 = 0$). From our sample calculation above, we already see that $H^*(K(\mathbb{Z},2);\mathbb{Q}) \cong \mathbb{Q}[x_n]$ and $H^*(K(\mathbb{Z},1);\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_n)$. Then to the fibration $K(\mathbb{Z},2) \to P(K(\mathbb{Z},3)) \to K(\mathbb{Z},3)$ we have the following Serre spectral sequence.



We can see that a similar argument as the one in the previous calculation gives us that $H^*(K(\mathbb{Z},3);\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_3)$. Then by induction using fibrations $K(\mathbb{Z},n-1) \to P(K(\mathbb{Z},n)) \to K(\mathbb{Z},n)$, we conclude our proposition. We can use a similar (simpler) technique to show that $H^*(K(G,n);\mathbb{Q})$ is trivial (i.e. $\mathbb{Q}$ only in dimension 0) for finite G. Combining these results, then using results from algebra we conclude that

$$H^k(K(\mathbb{Z},n)) \equiv H^k(K(\mathbb{Z},n);\mathbb{Z}) = \begin{cases} 0 \text{ if } 0 < k < n \\ \mathbb{Z} \text{ if } k = 0, n \\ \mathbb{Z} \oplus G_{\text{finite}} \text{ if } n|k,\ n \text{ even} \\ G_{\text{finite}} \text{ otherwise} \end{cases}$$

and $H^*(K(G,n))$ is trivial for finite G.

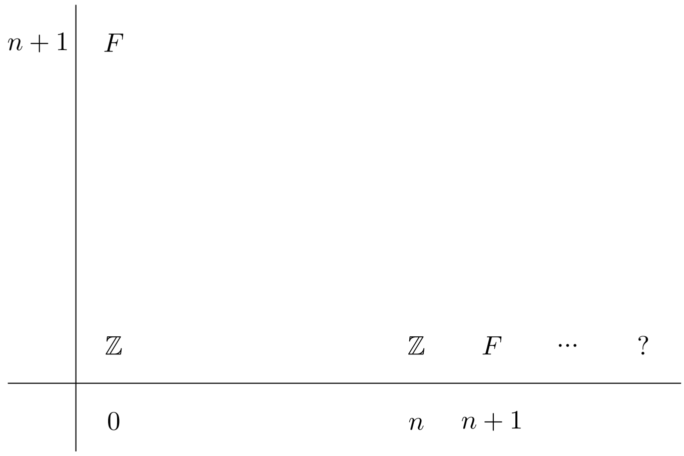
Now, recall the existence of the following construction:

Lemma 7.2. *Postnikov Towers. For any topological space X, for every n there exists a space P^nX such that $\pi_i(P^nX) \cong \pi_i(X)$ for $i \leq n$ and $\pi_i(P^nX) \cong 0$ otherwise.*

Now, to prove theorem 7.1, we will employ the Serre spectral sequence on the fibration

$$K(\pi_{n+1}S^n, n+1) \to P^{n+1}S^n \to P^nS^n$$

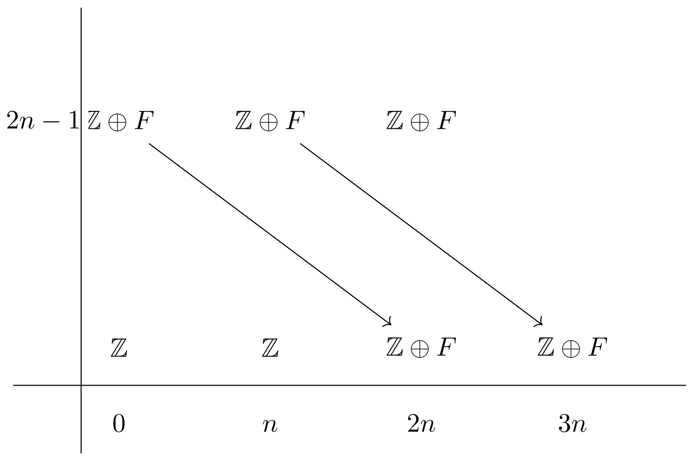
First, we recognise that P^nS^n has the same homotopy groups as $K(\mathbb{Z}, n)$ (i.e. they are weakly homotopy equivalent). We can deduce that $H^{n+1}(P^nS^n)$ is finite using a CW complex construction argument. Now, the corresponding Serre spectral sequence of this fibration for odd n, with F representing finite groups, is as follows



Here, we conclude that $H^{n+1}(K(\pi_{n+1}S^n, n+1)) = \pi_{n+1}S^n$ is finite. A similar argument works for the fibration

$$K(\pi_{n+k}S^n, n+k) \to P^{n+k}S^n \to P^{n+k-1}S^n$$

Now, for n even, we can deduce that the $2n$ page of our spectral sequence for the fibration $K(\pi_{n+1}S^n, n+1) \to P^{n+1}S^n \to P^nS^n$ is



Now the reader, with their new found knowledge of how to utilise spectral sequences, can deduce what happens for the fibration $K(\pi_{n+k}S^n, n+k) \to P^{n+k}S^n \to P^nS^n$ for even n. We have proved theorem 7.1.

8 The Adams Spectral Sequence

There are two fundamental objects in the construction Adams spectral sequence. These are **Spectra** and the **Steenrod Algebra**. Firstly, for our purposes, we need only to know that the Steenrod algebra $\mathcal{A}$ is a module generated by $\{Sq^{2^n} | n \in \mathbb{Z}_{\geq 0}\}$, subject to a complicated set of relations called the **Adem relations**. Most importantly these relations ensure that every element in $\mathcal{A}$ can be written as a linear combination of terms of the form $...Sq^{n_3}Sq^{n_2}Sq^{n_1}$ with $n_i \leq 2n_{i+1}$.

Definition 8.1. *A spectrum consists of a sequence of basepointed spaces X_n, $n \geq 0$, together with basepoint-preserving maps $\Sigma X_n \to X_{n+1}$.*

There are natural isomorphisms $H^m(X;G) \cong [X, K(G,m)]$ for all spectra X (recall the Eilenberg-Maclane definition of cohomology). The main point of this construction is to conveniently represent stable homotopy groups, so we define $\pi_i X_n = \lim_{n\to\infty} \pi_{n+i} X_n$. Here, let us review one of the core concepts from homological algebra:

Definition 8.2. *Let R be a ring and let A, B be groups. Then define $Ext^i_R(A,B)$ as follows. First, pick a* ***free resolution*** *of A over R. That is, an exact sequence $... \to F_1 \to F_0 \to A \to 0$ with the F_i being free R-modules. Then the ith cohomology group of the cochain complex $0 \to Hom_R(P_0, B) \to Hom_R(P_1, B) \to ...$ is $Ext^i_R(A,B)$.*

Now, associated to each spectrum is an **Adams spectral sequence**, that is a spectral sequence defined by the second page,

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(Y;\mathbb{Z}/2), H^*(X;\mathbb{Z}/2))$$

To make sense of this, observe that $\mathrm{Ext}^{s}_{\mathcal{A}}(H^*(Y;\mathbb{Z}/2), H^*(X;\mathbb{Z}/2))$ yields a graded ring $H^*(...)$, and t picks out the dimension of this result.

9 The Melody of the Spheres

Since this report is centered around understanding spheres in particular, we will focus on one particular spectrum of interest. That is of course the sphere spectrum

$$\mathbb{S} = S^0 \xrightarrow{\Sigma} S^1 \xrightarrow{\Sigma} S^2 \xrightarrow{\Sigma} ...$$

Letting $X = Y = \mathbb{S}$ in our definition of the Adams spectral sequence, we obtain

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

perhaps the most famous spectral sequence diagram of all time, colloquially known among topologists as the "musical score of the spheres".

Now, convergence in the Adams spectral sequence is reasonably analagous to convergence in the Serre sense.

Theorem 9.1. *For fixed s and t the groups* $E_r^{s,t}$ *are independent of r once r is sufficiently large, and the stable groups* $E_\infty^{s,t}$ *are isomorphic to the quotients* $F^{s,t}/F^{s+1,t+1}$ *for the filtration of* $\pi_{t-s}(\mathbb{S})$ *by the images* $F^{s,t}$ *of the maps* $\pi_t(S^s) \to \pi_{t-s}(\mathbb{S})$.

That is to say that the E_∞ page of our Adams spectral sequence reveals information about the homotopy groups of the spheres. In intuitive terms, this says that the number of groups on the $t - s = i$ diagonal which survive to the E_∞ page is the rank of the stable homotopy group $\pi_i(\mathbb{S})$.

9.1 Calculating the Free Resolution

Now, computing the terms in the Adams spectral sequence essentially boils down to finding a free resolution of $\mathbb{Z}/2$ over the Steenrod algebra $\mathcal{A}$. A complete calculation of this is still an open problem due to the complexity of the aforementioned Adem relations.

We wish to compute the free resolution ... $\xrightarrow{d_3} F_1 \xrightarrow{d_2} F_0 \xrightarrow{d_1} \mathcal{A} \xrightarrow{\varepsilon} \mathbb{Z}/2$. We can with the canonical map $\mathcal{A} \xrightarrow{\varepsilon} \mathbb{Z}/2$, where $\varepsilon(1) = 1$ and $\varepsilon(Sq^n) = 0$ for $n \geq 1$. Then by exactness, we naturally we obtain that $F_0 = \mathcal{A}\{x_1, x_2, x_4, x_8, ...\}$ where $d_1(x_i) = Sq^i$.

Calculating d_2 is significantly more complicated and makes extensive use of the Adem relations. Below are some common Adem relations.

$Sq^1Sq^{2n} = Sq^{2n+1}$	$Sq^3Sq^{4n} = Sq^{4n+3}$
$Sq^1Sq^{2n+1} = 0$	$Sq^3Sq^{4n+1} = Sq^{4n+2}Sq^1$
$Sq^2Sq^{4n} = Sq^{4n+2} + Sq^{4n+1}Sq^1$	$Sq^3Sq^{4n+2} = 0$
$Sq^2Sq^{4n+1} = Sq^{4n+2}Sq^1$	$Sq^3Sq^{4n+3} = Sq^{4n+5}Sq^1$
$Sq^2Sq^{4n+2} = Sq^{4n+3}Sq^1$	$Sq^4Sq^3 = Sq^5Sq^2$
$Sq^2Sq^{4n+3} = Sq^{4n+5} + Sq^{4n+4}Sq^1$	$Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2$

From this, we can for example deduce that $d_1(Sq^1(x_1)) = 0$, hence F_1 must contain a generator, say y_2, which d_2 maps to $Sq^1(x_1)$. We can also check that $d_2(Sq^2(y_2)) = Sq^2Sq^1(x_1)$, which is indeed in the kernel of d_1. The next generator y_4 in F_1 would map to $Sq^3(x_1) + Sq^2(x_2)$. On the following page is a table of further calculations for low dimensions.

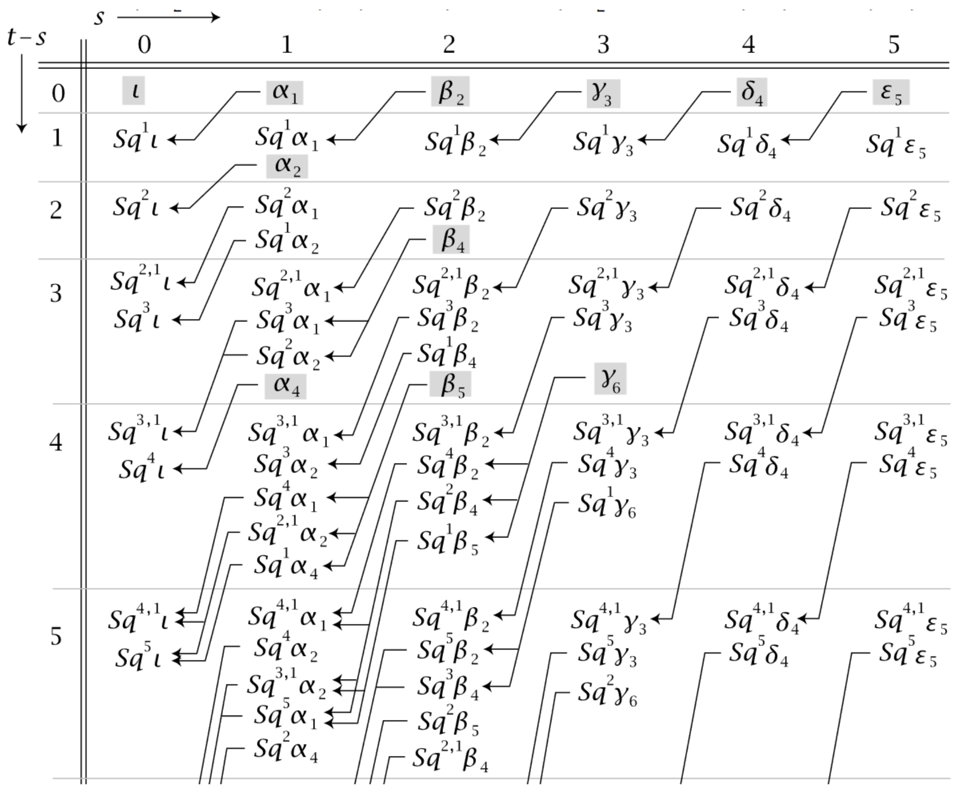


Figure 1: (Hatcher) Low dimension calculations of the free resolution of $\mathbb{Z}/2$ over $\mathcal{A}$

10 The Musical Score of the Spheres

Below is the diagram of the Adams spectral sequence. Note that the x-axis being labelled $t-s$ means that the columns give information about the stable homotopy groups (or equivalently the homotopy groups of spectra).

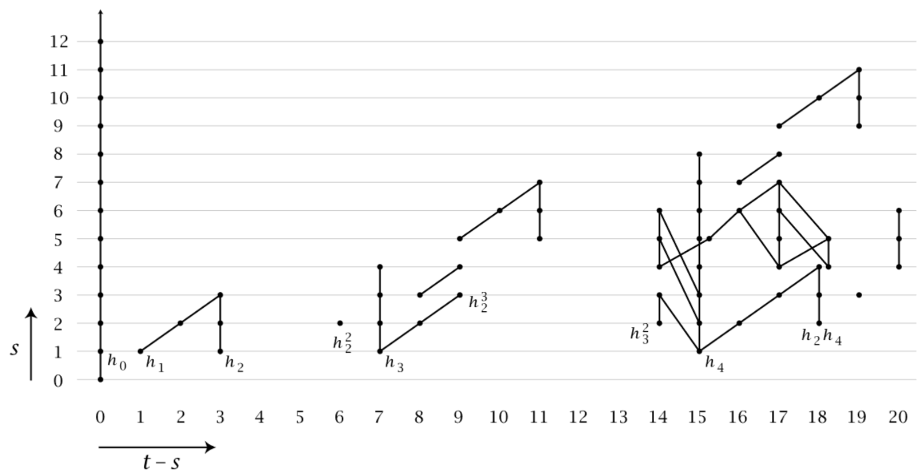


Figure 2: (Hatcher) The Adams spectral sequence

Most notably, the points labelled h_i correspond to sphere maps, and the differential from the point labelled h_4 (and subsequent differentials from points h_5 and above) were used to prove the non-existence of division algebras of dimensions other than $1, 2, 4$ or 8.

11 Discussion and Conclusion

We could just as easily have considered the Adams spectral sequence of a different spectrum of spaces other than the sphere spectrum. However, I have chosen to apply this powerful tool to understanding spheres since it is simply the most culturally significant application in modern mathematics. The Adams spectral sequence is a very deep topic, and this report certainly does not completely do it justice. A good majority of insights we can derive from the spectral sequence see application in differential geometry (much of which I do not personally understand). Using 21st century computational power, we have been able to fully compute the Adams spectral sequence up to around dimension 60, however the nature of the Adem relations make current algorithms grind to a halt around this limit. To develop a more efficient algorithm remains an open problem.

References

[1] Allen Hatcher. Spectral Sequences, https://pi.math.cornell.edu/ hatcher/AT/ATch5.pdf (2004).

[2] J. F. Adams, "On the Non-Existence of Elements of Hopf Invariant One." Annals of Mathematics, vol. 72, no. 1, Annals of Mathematics, 1960, pp. 20–104, https://doi.org/10.2307/1970147.

[3] J. Adem, The iteration of the Steenrod squares in algebraic topology. Proc. Nat. Acad. Sci. USA, 1952.