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Nonlocal Interactions in Physics and Biology

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Prelude

Abstract

This report introduces nonlocal interactions modelled with the fractional Laplacian. In particular, focus is on motivations and applications in physics and biology.

We first discuss the fractional Laplacian and how it differs from the Laplacian. We then build some intuition for its effect in terms of the fundamental solution. Derivation is shown through considering the nonlocal heat equation and its applications to anomalous transport. This is then broadened to a discussion on the Lévy flight hypothesis and the hunting patterns of animals, as well as modelling nonlocal populations. We discuss the similarities between the biological treatments and fractional quantum mechanics, and discuss possible future directions in relating nonlocality to self-organised criticality.

Acknowledgements

I would like to sincerely thank Serena Dipierro and Enrico Valdinoci for supervising me in this project. I feel extraordinarily lucky to have the support of world leaders in this field, and their discussions on mathematics and beyond are always insightful and pleasant.
I would also like to thank the AMSI Events team for this fantastic opportunity to engage in research as an undergraduate. In particular, meeting other like-minded mathematicians through AMSI Connect and learning of the areas of mathematics that other universities focus on was a very enjoyable and very well organised experience.

Statement of Authorship

Originality can be found in the form of a simple hypothetical and possible future direction in Sections 3.1 and 6. The bulk of this report is sourced from the works referenced, and from discussions with my supervisors. No new theorems are discussed in this report, and it is best to view this report as a literature review.

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1 Introduction

1.1 Why Nonlocal?

The Nobel prize in physics for 2022 was awarded to Aspect, Clauser and Zeilinger for work that included demonstrating and understanding the phenomena of quantum entanglement. I had the opportunity to replicate the violation of the Bell inequality, one of the key experiments in this field, in 2022 as part of a laboratory in physics. This experiment shows that pairs of entangled photons can exhibit what Einstein famously called 'spooky action at a distance': the phenomena in which one photon instantaneously affects its partner, no matter how far apart they are. Whether this is nonlocal or nonrealistic behaviour is up to your interpretation of quantum mechanics, but the experiment undeniably shows that we should consider how objects can affect each other even when they are far apart.

More generally that quantum entanglement, we see a rich world of nonlocality all around us. This report, I assume, has ended up on your computer despite the fact that I did not physically place it there. If I transmit this report across the internet then, on the time scale I care about, I find that I can instantaneously affect another computer. A wolf howling at night can let another wolf, many kilometres away, know its location almost as quickly as it could have let a wolf a few metres away know. An unknowing COVID-positive patient taking a flight across the world can in a matter of hours transmit something extremely powerful and frightening. On the time scales we care about, this may as well be instantaneous. In this way, we aren't so different from the photons.

In understanding the nonlocality of the world, we see that it is useful to model it. However, the mechanical nature of much of our mathematics, including traditional calculus, means we require the formulation of more powerful tools in the form fractional derivatives, which constitute nonlocal operators. In this report, we will discuss one such operator, the fractional Laplacian, and discuss some applications in physics and biology.

1.2 Scope

While an infinite family of nonlocal operators exist, this report will focus on the fractional Laplacian. While the Laplacian is generated by considering the diffusion of particles in Brownian motion, the fractional Laplacian is generated by considering particles that are allowed to jump anywhere on the domain. In this way, we see the nonlocality of the operator arise.

This nonlocality is extremely useful in modelling real-world phenomena. Of particular interest in this report is the use of the fractional Laplacian in modelling biological processes in the Lévy flight hypothesis, which states that optimal search strategies for scarce resources may arise from a Lévy flight. We also touch on interesting nonlocal physics in anomalous diffusion and the fractional Schrödinger equation. In addition to these applications that will be discussed, there is an ever growing number of fields that have recently acquired a nonlocal treatment. Some examples from physics include Fractional Lane-Embden equations for generalised star modelling (Chen 2018), crystal dislocations (Dipierro, Palatucci, and Valdinoci 2015), statistical mechanics (Cozzi, Dipierro, and Valdinoci 2017) and projectile motion with wind (Özarslan et al. 2020).

2 The Nonlocal (Fractional) Laplacian

2.1 The Classical Laplacian

Let us recall first the Laplacian of a function (indicated by the operator Δ), which we will refer to as the 'classical' Laplacian. This is most commonly defined as the sum of second derivatives:

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) \tag{1}$$

An alternative, somewhat surprising, but very elegant, definition lies in an integral formulation:

$$-\Delta u(x) = \lim_{r \searrow 0} \frac{\text{const}}{r^{n+2}} \int_{B_r(x)} (u(x) - u(y))dy \tag{2}$$

In this way we can see that the Laplacian can be thought of as an averaging function, where the Laplacian of a function at a point x considers the average difference between the function's value at x and at all the points in the neighbourhood.

2.2 The Fractional Laplacian

The fractional Laplacian, denoted by the operator $(-\Delta)^s$ with $s \in (0,1)$, is named such due to the property that

$$(-\Delta)^s(-\Delta)^{s'}u(x) = (-\Delta)^{s+s'}u(x) \tag{3}$$

The fractional Laplacian is defined in its integro-differential form, often called the 'Riesz' definition or the 'integral' definition in the literature, as follows:

$$(-\Delta)^s := -\frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \tag{4}$$

or, more conveniently, this can be represented (Abatangelo and Valdinoci 2019)

$$(-\Delta)^s = C(n,s)\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{5}$$

with

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos \omega_1}{|\omega|^{n+2s}} d\omega \right)^{-1} \tag{6}$$

The integral is defined in the Principal Value sense, as indicate by 'P.V.'. This is to account for the singularity when $x = y$, and is more accurately states

$$(-\Delta)^s u(x) := \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{7}$$

Other properties of note regarding the fractional nature are that

$$\lim_{s\to 1}(-\Delta)^s u(x) = -\Delta u(x) \tag{8}$$

$$\lim_{s\to 0}(-\Delta)^s u(x) = u(x) \tag{9}$$

which is essentially to say that $s \to 1$ recovers the classical Laplacian, while $s \to 0$ results in the operator acting trivially on the function.

2.3 Compare the Pair

Note two key differences between the Laplacian and fractional Laplacians' definitions.

Laplacian	Fractional Laplacian
$-\Delta u(x)$	$(-\Delta)^s u(x)$
$\lim_{r\searrow 0} \frac{\text{const}}{r^{n+2}} \int_{B_r(x)} (u(x) - u(y))dy$	$C(n,s)\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\lvert x-y\rvert^{n+2s}} dy$

First, the integration limits: the Laplacian is computed by integration over a ball that is sent to a radius of 0. In this way, the averaging property of the Laplacian is taken only over the points in the immediate neighbourhood of x. In comparison, the fractional Laplacian is integrated over all space. This point is why the fractional Laplacian is nonlocal, and hence is called the fractional Laplacian or nonlocal Laplacian interchangeably.
The second key difference is in how this averaging occurs. See that in the classical Laplacian, each point within $B_r(x)$ contributes equally and is averaged by dividing the integral by $\frac{1}{r^{n+2}}$. In the case of the fractional Laplacian, each point's contribution is weighted by the kernel $\frac{1}{\lvert x-y\rvert^{n+2s}}$ within the integrand, and so points close to x contribute more than points far away.

2.4 What about fractional derivatives?

It is tempting to attempt to define the fractional Laplacian in terms of fractional derivatives, where instead of the sum of second derivatives we have the sum of α derivatives, with $\alpha \in (0, 2)$. However, this is an inappropriate description. As a simple example of why, let us consider some fractional derivative $\frac{\partial^\alpha f}{\partial x^\alpha}$. The fractional nature requires that

$$\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^{2-\alpha} f(x)}{\partial x^{2-\alpha}} = \frac{\partial^2 f}{\partial x^2}. \tag{10}$$

Let us assume that we can construct the fractional Laplacian as

$$(\Delta)^{\alpha/2} f(x) = \sum_{i=1}^{n} \frac{\partial^\alpha}{\partial x_i^\alpha}(x) f(x) \tag{11}$$

For the purpose of a simple counterexample, suppose we choose $\alpha = 1$ such that the fractional derivatives turn to first derivatives, then we see for $n > 1$ dimensions, due to the chain rule,

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(x) \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f(x) \neq \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x) = \Delta f(x) \tag{12}$$

and hence (11) was a poor construction of the fractional Laplacian. I highlight this to emphasise the importance and beauty of the Riesz integral definition of the fractional Laplacian, that although it seems somewhat removed from the basic definition of the Laplacian in terms of second derivatives, it is arguably the most intuitive definition, as long as we understand the averaging, ‘democratic’, nature of the Laplacian in integral form.

2.5 Alternative Definitions

As Kwaśnicki (2017) summarises, there are at least ten different equivalent definitions of the fractional Laplacian. Of particular note is the Fourier definition, also known as the spectral definition, in the space $\mathscr{L}^p$, $p \in [1, 2]$, which appears to great utility in the literature:

$$\mathscr{F}\left((-\Delta)^s f(\xi)\right) = -|\xi|^{2s} \mathscr{F} f(\xi) \tag{13}$$

While this definition is equivalent to the integral formulation in $\mathbb{R}^n$, Lischke et al. (2020) discuss that over bounded domains there arises significant differences between the Fourier and Riesz definitions. As an example, see Figure 1. This difference is explained in terms of the regularity results for the two definitions, however is beyond the scope of this report. I will continue my discussions using the Riesz definition.

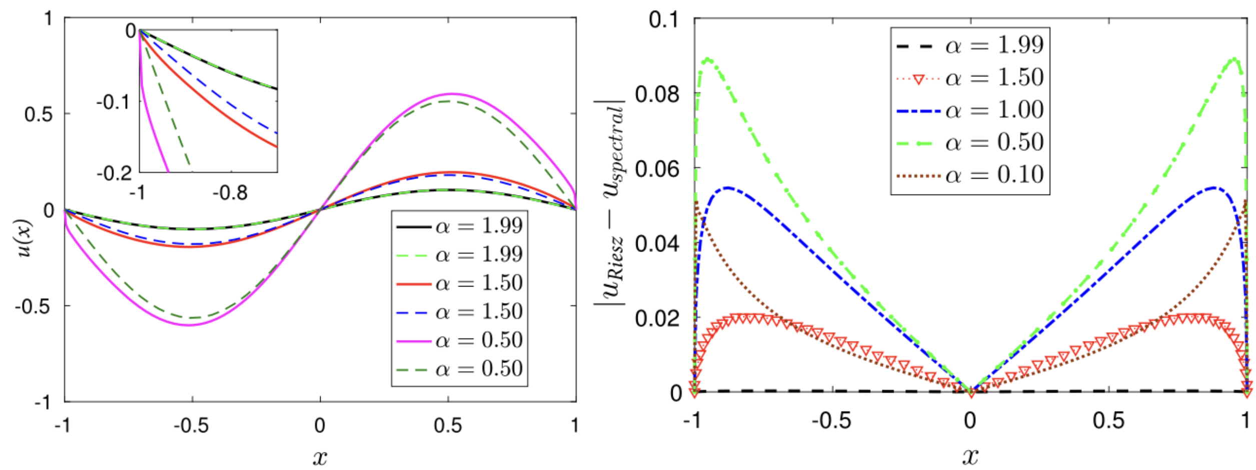


Figure 1: *Left*: numerical solutions to $(-\Delta)^s f(x) = \sin \pi x$ using Riesz definition (dashed lines) and Fourier definition (solid lines) of the fractional Laplacian. *Right*: Absolute differences between the two numerical solutions. This is computed in one-dimension over the interval $(-1, 1)$. These plots are from Lischke et al. (2020).

3 The Fundamental Solution

In this section, we seek to build some intuition about how basic functions change as a result of generalising from the classical Laplacian to the fractional Laplacian. We do this by considering the fundamental solution of the fractional Laplacian f, which is the solution to the equation

$$\Delta f = \delta_0(x) \tag{14}$$

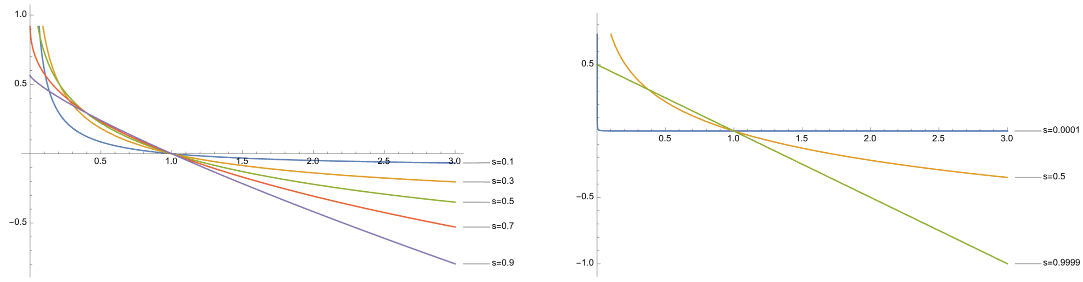
Where $\delta_0(x)$ is the Dirac delta function. Essentially, this is prescribing the only nonzero element of the Laplacian to being exactly on a source at the origin. This is useful, for instance, in $n = 3$ in order to describe the electrostatic or gravitational potentials due to a point charge/mass. As has been found by Abatangelo, Dipierro and Valdinocci in work that is still in progress, in the fractional case

$$(-\Delta)^s f(x) = \delta_0 \tag{15}$$

the fundamental solution is

$$\mathbb{R}^n \backslash \{0\} \ni x \mapsto f(x) := \begin{cases} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{2^{2s}\pi^{n/2}\Gamma(s)} \left(|x|^{2s-n} - 1\right) & \text{if } n \neq 2s \\ -\frac{1}{\pi} \ln |x| & \text{if } n = 1 \text{ and } s = \frac{1}{2} \end{cases} \tag{16}$$

For example, see Figure 2, in which we compare one dimensional instances of the above equations (i.e. $n = 1$). In Fig 2a, we see an overview of how the fundamental solution changes with s. Perhaps more interesting is Fig 2b, where we see that as $s \to 0$, we find that the fractional Laplacian acts trivially, as we know from Section 2.2, and so the fundamental solution resembles the Dirac delta distribution. As $s \to 1$, we recover the classical fundamental solution, which in $\mathbb{R}$ is $f(x) \sim |x|^{-1}$. In a hand-waving way, we can compare the curves in Fig 2a and see that varying s is balances the fundamental solution between these two extremes of either the fractional Laplacian not doing anything, or recovering the classical fundamental solution.



(a) (b)

Figure 2: Fundamental solution of the fractional Laplacian for different values s.

3.1 A Simple Thought on Nonlocal Gravity

From this understanding of the fundamental solution, a fun hypothetical we can ask is how gravitational attraction varies with the fractional exponent s. To this end, we can suppose the Earth's mass of 5.972×10^24kg is a point mass at the Earth's centre, and that we are standing at the surface of the Earth, about 6.378×10^6m from the centre. Using Newton's laws of gravitational attraction, we can recall that the force due to gravity is given

$$F = -\nabla V \tag{17}$$

Where

$$V(r) = \frac{-GMm}{r} \tag{18}$$

is the gravitational potential at radius r for the Earth's mass M, our mass m and Newton's Gravitational constant G. Now if we (quite naively) create a nonlocal gravitational potential V_s based on the fundamental solution, we might like to define it as

$$V_s(x) := GMm|r|^{2s-3} \tag{19}$$

which is constructed from the fundamental solution, with constants adjusted, $n = 3$, and a vertical translation. Now, we can continue by differentiating and using $F = ma$ to see that the acceleration at the surface of the Earth in this nonlocal regime would be given

$$a_s = GM(2s-3)|r|^{2s-4} \tag{20}$$

Which is displayed in Figure 3.

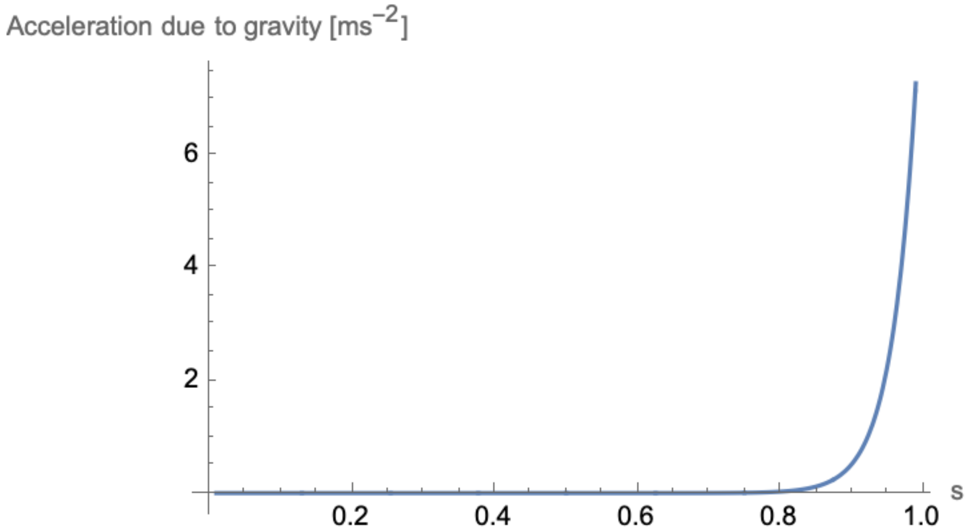


Figure 3: Acceleration due to nonlocal gravity at the surface of the Earth. $a \approx 9.8\text{ms}^{-2}$ as $s \to 1$.

Notice that very quickly as the fractional exponent decreases, objects would start getting very light. I will note, too, that one reason why this is probably a terrible model for nonlocal gravity is that it doesn't account for actual nonlocality in the physical system. We treated the Earth as a point mass because we are allowed to in the classical case, but perhaps if gravity did act nonlocally in this hypothetical, then contributions from the other side of the Earth would act differently compared to local contributions. Perhaps, then, this model only really works if you are far away from the object and so can treat it as a point mass. Also, this model uses ms^{-2} in Figure 3, but the fractional equations actually alter the units themselves, although this would be fixed by the introduction of G_s, a nonlocal gravitational constant.

4 Motivating the fractional Laplacian

4.1 The Nonlocal Heat Equation

A convenient and intuitive motivation for the fractional Laplacian arises from considering the heat equation. This subsection is a rewriting of work outlined in Abatangelo and Valdinoci (2019), Valdinoci (2009) and Bucur, Valdinoci, et al. (2016).

4.1.1 The Classical Heat Equation

Perhaps the most famous partial differential equation is the heat equation:

$$\partial_t u(x,t) = c\Delta u(x,t) \tag{21}$$

where $u(x,t)$ prototypically represents temperature, but of course the use of this equation extends far beyond. A derivation of the heat equation comes from considering discrete random walks, in which at some time step τ, we have some small spatial interval $h > 0$ such that

$$\tau = h^2 \tag{22}$$

The random walk takes place on a lattice $h\mathbb{Z}^n$. Let us place a particle at some point on the lattice, and allow it to travel, at each time step, to a neighbouring coordinate with equal probability. For example, in $\mathbb{Z}^2$, the particle can move left, right, up, or down (see Figure 4a). In $\mathbb{Z}^3$, the particle can additionally move backwards and forwards, and so on for more dimensions. The derivation of the heat equation is shown in Appendix A

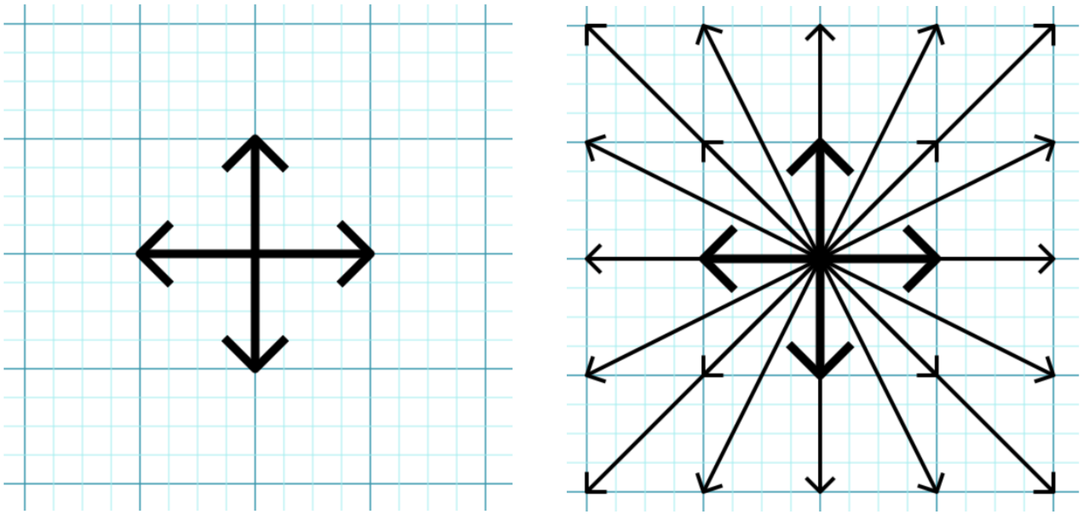
Physically, the link between the particle's random walk and the heat equation can be thought of as a diffusion process. Imagine, for example, heating one particular part of a block of metal to an extremely high temperature. On the atomic scale, heat transfer is indeed achieved by the random movements of particles colliding with other particles and transferring kinetic energy. Thus, the macroscopic diffusion of heat is explained by analysing these microscopic random walks.

4.1.2 The Nonlocal Heat Equation

The generalisation to the nonlocal heat equation is essentially through reasoning, why should a particle on be allowed to walk to its neighbouring spots? Why should it not be allowed to jump far away, with the probability of it doing so diminishing as the jump length increases? This is illustrated in Figure 4b. On some allowed jump region described by the domain $\Omega \in \mathbb{R}^n$, the particle can travel anywhere with probability of doing so inversely proportional to the length of the jump. Through this treatment, we derive the nonlocal heat equation

$$\partial_t u(x,t) = c(-\Delta)^s u(x,t) \tag{23}$$

which is proved in Appendix B. This is one intuitive motivation for the fractional Laplacian, as we can see that it describes the nonlocal diffusion of particles, whereas the classical Laplacian describes the case when particles can only travel in their immediate neighbourhood.



(a) In a discrete random walk, the particle may move to any neighbouring coordinate with equal probability.

(b) In the nonlocal regime, the particle may jump to any coordinate in the prescribed domain, with its probability of doing so being inversely proportional to the length of the jump.

Figure 4

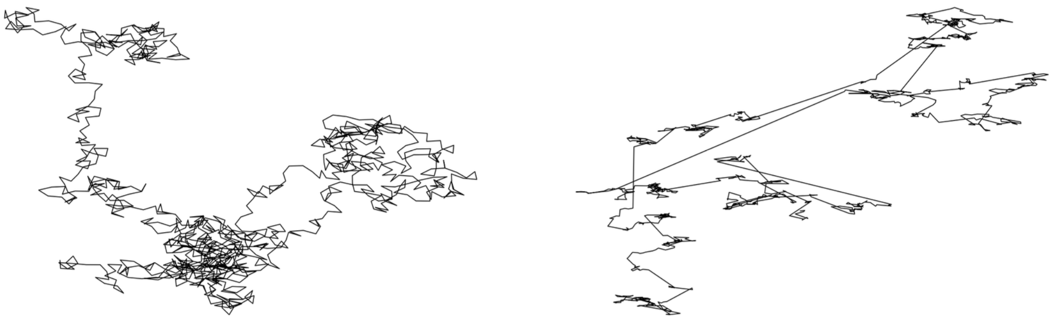
4.2 Anomalous Transport

If the classical heat equation describes a standard diffusion process, then the nonlocal heat equation describes what is called 'anomalous' diffusion. This kind of diffusion is observed in soft matter (McKinley, Yao, and Forest 2009), plasmas, glassy materials (Klages, Radons, and Sokolov 2008), and in COVID transmission (Akindeinde et al. 2022). The modelling of anomalous transport in the literature utilises a wide range of fractional operators, but essentially the benefit of this treatment is that fractional operators are nonlocal, which allow consideration of long-range effects. In the case fluid dynamics, such as that observed in plasmas, these effects are things like turbulence and drag. In disease modelling, a time-fractional operator is used to allow equations to take into account things that have happened in the past. For example, it is a good idea to build into the model some kind of immunity period after catching COVID, and the fractional operator allows this. We can think of this as being nonlocal in time.

5 Nonlocal Biology

5.1 The Lévy Flight

Instead of diffusion, another way of viewing the random movement of a particle in the classical case is as a simple random walk. In the limit as the lattice and the time steps shrink, we recover Brownian motion, as seen in Figure 5a. However, if we permit the particle to take these long jumps, with jump-length probability following a power law distribution, then the particle follows the Lévy flight (Metzler and Klafter 2000), as seen in Figure 5b. Viswanathan et al. (1996) observed that albatrosses appear to follow a Lévy distribution in the



(a) Brownian motion

(b) Lévy flight ($s = 0.75$)

Figure 5: Random walks

length of their flights, and there appears to be good reason for this. The Lévy flight hypothesis argues that when food is scarce, it is best to follow a Lévy flight rather than Brownian motion when searching for food (Klages 2018). This makes sense intuitively, as circling the same area will not yield the albatross many fish, especially if it dives and scares the fish away. Instead, the albatross should search an area and, if it doesn't find many fish there, then fly far away and search that new area.

The authors of the 1996 seminal paper outlining the albatross' flights revisited the data and found that it actually disagreed with their original conclusion (Edwards et al. 2007). It is argued that perhaps the sum of exponential distributions, or a Gamma distribution, suits the data better. This statistical question arises from the difficulty of dealing with heavy tail distributions, who appear quite similar, especially with low sample sizes. Nonetheless, in the two and a half decades following the work of Viswanathan et al., many animals have been observed to follow the Lévy flight (Benhamou 2007). One notable example of the Lévy flight is in crime modelling (Chaturapruek et al. 2013). It seems that burglaries follow a Lévy distribution, as once one house is broken into, that same house or houses in the neighbourhood have an increase chance of being burgled in the coming weeks. The robbers are in the area, and they have learned the neighbourhood, after all. After some weeks, perhaps once people start locking their windows and installing security systems, the burglars move on, and hence exhibit the long jump behaviour characteristic of the Lévy flight. It is fascinating, then, that the nonlocal diffusion that was discussed in Section 4 in the context of physics has applications in biology and modelling human behaviour.

5.2 Efficient Populations

If we understand that biological populations often follow Lévy flights, it motivates the modelling of populations in terms of nonlocal migrations. One such model comes from adapting the commonly used logistic equation.

The discussions in this subsection are adapted from the work of Caffarelli, Dipierro, and Valdinoci (2016) and Dipierro, Savin, and Valdinoci (2017).

5.2.1 The Classical Case

$$\partial_t u = \Delta u + (\rho - u)u \tag{24}$$

Where $u = u(x,t)$ is the population density term and $\rho = \rho(x,t)$ is some term indicating the maximum sustainable population. $\partial_t u$ accounts for births and deaths, while Δu, in the spirit of the heat equation, describes the migration of animals within the domain. $(\rho - u)u$ is a term that accounts for resources where, for example, if there is overpopulation then the term is negative as animals starve, and if u is somewhere between 0 and ρ then there is population growth as there is more food than they are eating.

One possible case of interest is when the system is time-stationary, i.e. $\partial_t u = 0$ as births and deaths are in perfect balance, and so

$$-\Delta u = (\rho - u)u \tag{25}$$

And then we may want to ask what an efficient population looks like, where $\rho = u$, so that the population is maximising the use of resources everywhere. The equation reduces to Laplace's equation, as we are asking for solutions where

$$\rho = u \tag{26}$$

$$\Delta u = 0 \tag{27}$$

The trouble here is that harmonic functions (functions satisfying Laplace's equation) are somewhat rare, so for arbitrary ρ there is unlikely to be a solution to (27). Hence, in the classical case, we simply cannot find in the general case a solution for efficient populations.

5.2.2 The Nonlocal Case

Understanding the migration of animals as a nonlocal process, we can replace the Laplacian in (24) with the nonlocal Laplacian. Then our nonlocal logistic equation becomes

$$\partial_t u = (-\Delta)^s u + (\rho - u)u \tag{28}$$

We ask the same questions, again, about a time-stationary solution where the population maximises the use of resources, so that

$$\rho = u \tag{29}$$

$$(-\Delta)^s u = 0 \tag{30}$$

But now we can exploit a beautiful result due to Dipierro, Savin, and Valdinoci (2017) that all functions are locally s-harmonic up to a small error. More specifically, for any $\epsilon > 0$, any $k \in \mathbb{N}$, and any function $u \in C^k(\bar{B}_1)$, there exists v_ϵ such that

$$\begin{cases} ||u - v_\epsilon||_{C^k B_1} \leq \epsilon \\ (-\Delta)^s v_\epsilon = 0 \text{ in } B_1 \end{cases} \tag{31}$$

For reference, the C^k norm of a function is defined

$$||f||_{C^k(\Omega)} := \sum_{i=0^k} \sup_{x \in \Omega} |f^{(i)}(x)| \tag{32}$$

The statement in (31) first describes that the proxy-function v_ϵ can be arbitrarily close to the original function u, and then that within B_1, the function is s-harmonic. The fact that this domain is in B_1 centred at the origin is no issue, as any function through translation and scaling can satisfy this.

The proof of this theorem is very involved, however the general overview is as follows:

1. Represent u as a polynomial, which is by definition the sum of monomial terms. The fractional Laplacian is linear (as $(-\Delta)^s(u + \alpha v) = (-\Delta)^s u + (-\alpha\Delta)^s v$). Therefore, the proof requires finding s-harmonic function for any $x^\beta/\beta!$ in $C^k(B_1)$.
2. Construct an s-harmonic function with the same derivatives as $x^\beta/\beta!$ up to order $|\beta|$ at the origin.
3. Rescale the this constructed s-harmonic function such that higher order terms (between $|\beta| + 1$ and k) vanish. This allows arbitrarily close approximation of $x^\beta/\beta!$.

This theorem means that when seeking solutions to (30), as it turns out, for arbitrary ρ we can always find a population distribution that is sufficiently close to this resource distribution such that the population is maximally efficient at using its environment. The key to why this is allowed in the nonlocal case but not the classical case is in the oscillation of $u(x, t)$ outside of the domain. If ρ is particularly nasty and makes it seemingly difficult to construct an s-harmonic function, then all that is required is oscillations off to infinity that counterbalance contributing terms within the domain. What is the physical meaning of this in terms of animal populations? Maybe hunter-gathering? Agriculture that does not need to be maximally efficient outside of a big city that does? Perhaps there is nice utilisation of this fact in the future for real population modelling, but for now we can be satisfied with appreciating the mathematics.

5.3 Fractional Quantum Mechanics

I've included fractional quantum mechanics in the nonlocal biology section because I think it is fitting to liken a quantum object to the albatrosses. More specifically, the standard Schrödinger equation (in this section, I'll move away from using the term 'classical' when referring to the non-fractional concepts) is derived from considering the Feynman path integral over quantum paths in Brownian motion (Dávila et al. 2015). This is not dissimilar from the heat equation. The result is the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{-\hbar^2}{2m}\Delta\Psi + V\Psi \tag{33}$$

With $i^2 = -1$, $\hbar$ is Planck's constant and $\Psi = \Psi(x,t)$ is the wavefunction.
Now, like an albatross, we will allow the quantum object to move with a Lévy flight instead of with Brownian motion and take the Feynman path integral. The result is the fractional Schrödinger equation (Jeng et al. 2010):

$$i\hbar\frac{\partial}{\partial t}\Psi = D_\alpha \hbar^{2s}(-\Delta)^s\Psi + V\Psi \tag{34}$$

where D_α is the fractional diffusion coefficient.

The study of solutions to the fractional Schrödinger equation is an emerging field, with the equation only being introduced this century (Laskin 2000). The intricacies of these is unfortunately beyond the scope of this report, however it is interesting to note this as the final example of the applications of the fractional Laplacian. The study of nonlocal operators is truly at the nexus of many diverse and fascinating fields.

6 Future Directions

One idea arising from this project that can potentially be explored further is the link between self-organised criticality (SOC) and nonlocality. SOC describes how the dynamics of a system organises around a critical point. In particular, it appears that power-law scaling is the signature of both of these phenomena; it is scale-invariant, just as Lévy flights are. One such recent example of SOC is shown by Korchinski et al. (2021), who suggest that neurons display criticality through 'neuronal avalanches', where the effect of electric activity is observed to follow a power law distribution in how far it reaches. Is this not just the same as the nonlocal particle who can jump anywhere, with the probability of jumping to a spot inversely proportional to the length of that jump? Nonlocal activity in the brain is already an established phenomena (Dipierro and Valdinoci 2018), so it is not too far a leap to consider that these neuronal avalanches may be a nonlocal activity.

7 Conclusion

The game we played with the fundamental solution, the heat equation, biological modelling and the fractional Schrödinger equation is to replace local operators with nonlocal ones, in particular replacing the Laplacian with the fractional Laplacian. This generalisation allows these mathematical models to incorporate nonlocality into their operation. Nonlocality is motivated by the simple idea that particles need not only follow a Brownian walk, but can jump anywhere on the domain. Nonlocal mathematics is fairly recent in its formulation, and current active areas of research include in particular problems like the fractional Schrödinger equation, or in more broad areas such as nonlocal minimal surfaces. While very new and exciting, the importance of nonlocality through fractional calculus was foreshadowed many years ago, when Leibniz wrote to L'Hôpital in September of 1695 about the half-derivative: 'It will lead to a paradox, from which one day useful consequences will be drawn'. It seems this day is here, and the consequences are indeed both useful and beautiful.

References

Abatangelo, Nicola and Enrico Valdinoci (2019). "Getting acquainted with the fractional Laplacian". In: *Contemporary research in elliptic PDEs and related topics*, pp. 1–105.

Akindeinde, Saheed O et al. (2022). "Caputo fractional-order SEIRP model for COVID-19 Pandemic". In: *Alexandria Engineering Journal* 61.1, pp. 829–845.

Benhamou, Simon (2007). "How many animals really do the Lévy walk?" In: *Ecology* 88.8, pp. 1962–1969.

Bucur, Claudia, Enrico Valdinoci, et al. (2016). *Nonlocal diffusion and applications*. Vol. 20. Springer.

Caffarelli, Luis, Serena Dipierro, and Enrico Valdinoci (2016). "A logistic equation with nonlocal interactions". In: *arXiv preprint arXiv:1601.05552*.

Chaturapruek, Sorathan et al. (2013). "Crime modeling with Lévy flights". In: *SIAM Journal on Applied Mathematics* 73.4, pp. 1703–1720.

Chen, Huyuan (2018). "Liouville theorem for the fractional Lane–Emden equation in an unbounded domain". In: *Journal de Mathématiques Pures et Appliquées* 111, pp. 21–46.

Cozzi, Matteo, Serena Dipierro, and Enrico Valdinoci (2017). "Planelike interfaces in long-range Ising models and connections with nonlocal minimal surfaces". In: *Journal of Statistical Physics* 167, pp. 1401–1451.

Dávila, Juan et al. (2015). "Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum". In: *Analysis & PDE* 8.5, pp. 1165–1235.

Dipierro, Serena, Giampiero Palatucci, and Enrico Valdinoci (2015). "Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting". In: *Communications in Mathematical Physics* 333, pp. 1061–1105.

Dipierro, Serena, Ovidiu Savin, and Enrico Valdinoci (2017). "All functions are locally s-harmonic up to a small error". In: *Journal of the European Mathematical Society* 19.4, pp. 957–966.

Dipierro, Serena and Enrico Valdinoci (2018). "A simple mathematical model inspired by the Purkinje cells: from delayed travelling waves to fractional diffusion". In: *Bulletin of mathematical biology* 80, pp. 1849–1870.

Edwards, Andrew M et al. (2007). "Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer". In: *Nature* 449.7165, pp. 1044–1048.

Jeng, M et al. (2010). "On the nonlocality of the fractional Schrödinger equation". In: *Journal of mathematical physics* 51.6, p. 062102.

Klages, Rainer (2018). "Search for food of birds, fish and insects". In: *Diffusive Spreading in Nature, Technology and Society*, pp. 49–69.

Klages, Rainer, Günter Radons, and Igor Mihajlovič Sokolov (2008). *Anomalous transport*. Wiley Online Library.

Korchinski, Daniel J et al. (2021). "Criticality in spreading processes without timescale separation and the critical brain hypothesis". In: *Physical Review X* 11.2, p. 021059.

Kwaśnicki, Mateusz (2017). "Ten equivalent definitions of the fractional Laplace operator". In: *Fractional Calculus and Applied Analysis* 20.1, pp. 7–51.

Laskin, Nikolai (2000). "Fractional quantum mechanics and Lévy path integrals". In: *Physics Letters A* 268.4-6, pp. 298–305.

Lischke, Anna et al. (2020). "What is the fractional Laplacian? A comparative review with new results". In: *Journal of Computational Physics* 404, p. 109009.

McKinley, Scott A, Lingxing Yao, and M Gregory Forest (2009). "Transient anomalous diffusion of tracer particles in soft matter". In: *Journal of Rheology* 53.6, pp. 1487–1506.

Metzler, Ralf and Joseph Klafter (2000). "The random walk's guide to anomalous diffusion: a fractional dynamics approach". In: *Physics reports* 339.1, pp. 1–77.

Özarslan, Ramazan et al. (2020). "Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel". In.

Valdinoci, Enrico (2009). "From the long jump random walk to the fractional Laplacian". In: *arXiv preprint arXiv:0901.3261*.

Viswanathan, Gandhimohan M et al. (1996). "Lévy flight search patterns of wandering albatrosses". In: *Nature* 381.6581, pp. 413–415.

A Derivation of the Heat Equation

A particle originally at some point $h\bar{k}$ with $\bar{k} \in \mathbb{Z}^n$ at time t has $2n$ options to which it can walk in the following time step. These $2n$ options are each of the lattice points $h\bar{k}+he_i$ with e_i the standard Euclidean basis of $\mathbb{R}^n$.

The probability of the particle being at $x \in h\mathbb{Z}^n$ at time $t \in \tau\mathbb{N}$ is denoted $u(x,t)$ and we can write that

$$u(x,t+\tau) = \frac{1}{2n}\sum_{i=1}^{n}(u(x+he_i,t)+u(x-he_i,t)) \tag{35}$$

since the particle ending up at x is dependent on it being at a neighbouring point in the previous time step, and then jumping to x.

Noting by Taylor approximation that

$$\begin{aligned}
&u(x+he_i,t)+u(x-he_i,t)-2u(x,t)\\
&=\left(u(x,t)+h\nabla u(x,t)\cdot e_i+\frac{h^2D^2u(x,t)e_i\cdot e_i}{2}\right)+\left(u(x,t)-h\nabla u(x,t)\cdot e_i+\frac{h^2D^2u(x,t)e_i\cdot e_i}{2}\right)-2u(x,t)+O(h^3)\\
&=h^2\partial^2_{x_i}u(x,t)+O(h^3) \qquad (36)
\end{aligned}$$

Also, subtracting $u(x,t)$ from both sides in (35) yields

$$u(x,t+\tau)-u(x,t) = \frac{1}{2n}\sum_{i=1}^{n}(u(x+he_i,t)+u(x-he_i,t)-2u(x,t)) \tag{37}$$

So now we can construct the time derivative $\partial_t u(x,t)$ by dividing by τ and sending it to 0, and we see

$$\begin{aligned}
\partial_t u(x,t) &= \lim_{\tau\searrow 0}\frac{u(x,t+\tau)-u(x,t)}{\tau}\\
&=\lim_{h\searrow 0}\frac{1}{2n}\sum_{i=1}^{n}\frac{u(x+he_i)+u(x-he_i)-2u(x,t)}{h^2} \text{ since } \tau=h^2\\
&=\lim_{h\searrow 0}\frac{1}{2n}\sum_{i=1}^{n}\partial^2_{x_i}u(x,t)+O(h) \text{ using (36)}\\
&=\frac{1}{2n}\Delta u(x,t) \qquad (38)
\end{aligned}$$

and we have recovered the heat equation.

B Derivation of the Nonlocal Heat Equation

Once again, the particle inhabits points on the lattice $h\mathbb{Z}^n$, this time with

$$\tau := h^{2s} \tag{39}$$

Start the particle at the point $h\bar{k} \in \Omega$ and describe the probability of it jumping to the point $hk \neq h\bar{k}$ as

$$P_h(\bar{k},k) := \frac{\chi_\Omega(h\bar{k})\chi_\Omega(hk)}{C|k-\bar{k}|^{n+2s}} \tag{40}$$

Where

$$\chi_\Omega(x) := \begin{cases} 1 \text{ if } x \in \Omega \\ 0 \text{ if } x \notin \Omega \end{cases} \tag{41}$$

and forces the particle to only jump within the region Ω. The normalisation constant $C > 0$ is defined

$$C := \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|k|^{n+2s}} \tag{42}$$

Also define for convenience the probability of the particle moving off its current position

$$c_h(\bar{k}) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} P_h(\bar{k}, k) \tag{43}$$

and the probability of the particle staying on its current position

$$p_h(\bar{k}) := 1 - c_h(\bar{k}) \tag{44}$$

As with the classical heat equation, let $u(x,t)$ describe the probability density that a particle is located at $x \in \Omega \cap (h\mathbb{Z}^n)$ at time $t \in \tau\mathbb{N}$.

Since the labelling of points on the lattice is arbitrary, let's suppose $x = 0 \in \Omega$ and thus $c_h := c_h(0)$ and $p_h = p_h(0)$. Then the probability of being back at the origin at time $t + \tau$ is a sum over the probability of it being anywhere else, and then jumping back to the origin, or it being at the origin and then staying at the origin. This can be written

$$u(0, t+\tau) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} u(hk, t) P_h(k, 0) + u(0,t)(1 - c_h) \tag{45}$$

Subtracting $u(0,t)$ from both sides and rearranging, see that

$$\begin{aligned} u(0, t+\tau) - u(0,t) &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} u(hk, t) P_h(k, 0) - u(0,t) c_h \\ &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(u(hk,t) - u(0,t)\right) P_h(k,0) \text{ using } (43) \\ &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(u(hk,t) - u(0,t)\right) \frac{\chi_\Omega(hk)}{C|k|^{n+2s}} \text{ using } (40) \\ &= \frac{h^{n+2s}}{C} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(u(hk,t) - u(0,t)\right) \frac{\chi_\Omega(hk)}{|hk|^{n+2s}} \end{aligned} \tag{46}$$

Now dividing by τ an recalling that $\tau = h^{2s}$, we see that

$$C \frac{u(0,t+\tau) - u(0,t)}{\tau} = h^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(u(hk,t) - u(0,t)\right) \frac{\chi_\Omega(hk)}{|hk|^{n+2s}} \tag{47}$$

Adding both sides of the equation to themselves, but noticing in the addition to the right hand side that we can reflect k to $-k$ due to the symmetry in (40), see that

$$2C \frac{h(0,t+\tau) - u(0,t)}{\tau} = h^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{\left(u(hk,t) - u(0,t)\right) \chi_\Omega(hk) + \left(u(-hk,t) - u(0,t)\right) \chi_\Omega(-hk)}{|hk|^{n+2s}} \tag{48}$$

Let's now define

$$\varphi(y) := \frac{(u(y,t)-u(0,t))\chi_\Omega(y) + (u(-y,t)-u(0,t))\chi_\Omega(-y)}{|y|^{n+2s}} \tag{49}$$

and notice that assuming u is smooth, then for y close to 0, utilising Taylor approximation

$$\begin{aligned} &|(u(y,t)-u(0,t))\chi_\Omega(y) + (u(-y,t)-u(0,t))\chi_\Omega(-y)| \\ =&|(u(y,t)-u(0,t)) + (u(-y,t)-u(0,t))| \\ =&\left|\nabla u(0,t)y + O(|y|^2) + (-\nabla u(0,t)y + O(|y|^2))\right| \\ =&O(|y|^2) \end{aligned} \tag{50}$$

Hence when y is close to 0,

$$\varphi(y) = O(|y|^{2-n-2s}) \tag{51}$$

Now taking the integral of $\varphi(y)$ (assuming it is bounded and Riemann integrable) and using the Riemann sum representation, see that

$$\int_{\mathbb{R}^n\setminus B_\delta} \varphi(y) = \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n} \varphi(hk)\chi^n_{\mathbb{R}}\setminus B_\delta(hk) = \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk)\chi_{\mathbb{R}^n\setminus B_\delta}(hk) \tag{52}$$

for fixed $\delta > 0$.

Also, see that for small δ, using (51)

$$\int_{B_\delta} \varphi(y)dy = \int_{B_\delta} O(|y|^{2-n-2s})dy = O(\delta^{2-2s}) \tag{53}$$

So using these two equations integrating $\varphi(y)$ over the two domains, we can sum the two to achieve integration over $\mathbb{R}^n$:

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(y)dy &= O(\delta^{2-2s}) + \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk)\chi_{\mathbb{R}^n\setminus B_\delta}(hk) \\ &= O(\delta^{2-2s}) + \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk) + \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk)(\chi_{\mathbb{R}^n\setminus B_\delta}(hk) - 1) \end{aligned} \tag{54}$$

Noticing now that when $h|k| < \delta$, and hence the particle is inside B_δ,

$$\begin{aligned} \left|\sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk)(\chi_{\mathbb{R}^n\setminus B_\delta}(hk) - 1)\right| &= \left|\sum_{k\in\mathbb{Z}^n, 0<h|k|<\delta} \varphi(hk)\right| \\ &\leq \text{const } h^n \sum_{k\in\mathbb{Z}^n, 0<h|k|<\delta} |hk|^{2-n-2s} \\ &= \text{const } h^{2-2s} \sum_{k\in\mathbb{Z}^n, 0<h|k|<\delta} \frac{|k|^{1-s}}{|k|^{n+s-1}} \\ &\leq \text{const } h^{2-2s} \left(\frac{\delta}{h}\right)^{1-s} \sum_{k\in\mathbb{Z}^n, 1\leq h|k|<\delta} \frac{1}{|k|^{n+s-1}} \\ &\leq \text{const } h^{2-2s} \left(\frac{\delta}{h}\right)^{1-s} \left(\frac{\delta}{h}\right)^{1-s} \\ &= \text{const } \delta^{2-2s} \end{aligned} \tag{55}$$

Which can be absorbed into the $O(\delta^{2-2s})$ term in (54) and hence we find

$$\int_{\mathbb{R}^n} \varphi(y)dy = O(\delta^{2-2s}) + \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk) \tag{56}$$

And now choosing δ to be small, we see the $O(\delta^{2-2s})$ term vanishes and we are left with

$$\int_{\mathbb{R}^n} \varphi(y)dy = \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk) \tag{57}$$

Finally, using (48) and observing we can take the limit as $t \searrow 0$ to recover the time derivative, we see

$$2C\partial_t u(0,t) = \lim_{h\searrow 0} 2C\frac{u(0,t+\tau) - u(0,t)}{\tau} \tag{58}$$

$$= \lim_{h\searrow 0} h^n \sum_{k\in\mathbb{Z}^n, k\neq 0} \varphi(hk) \tag{59}$$

$$= \int_{\mathbb{R}^n} \varphi(y)dy \tag{60}$$

$$= \int_{\mathbb{R}^n} \frac{(u(y,t) - u(0,t))\chi_\Omega(y) + (u(-y,t) - u(0,t))\chi_\Omega(-y)}{|y|^{n+2s}} dy \tag{61}$$

$$= -2(-\Delta)^s_\Omega u(x,0) \tag{62}$$

using the fractional Laplacian definition introduced in (4). The subscript Ω on the fractional Laplacian here is to indicate that this is defined on the domain Ω.