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Supersymmetric Quantum Mechanics and Witten Index

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Contents

1 Prelude

1.1 Acknowledgements

I want to thank Dr. Johanna Knapp for her supervision and assistance throughout the project. I feel very lucky to join her project.

1.2 Abstract

We explore a basic topological characteristic associated with supersymmetric theories, called the Witten index. The project starts with recalling basic concepts of classical mechanics including Lagrangian and action. Then this is extended to quantum mechanics, in which we construct a supersymmetrised version of a specific model.

From there we discover some characteristic properties of supersymmetric theories including conserved charges, the Hamiltonian, and a $\mathbb{Z}_2$ grading. Then the concepts of supersymmetric ground states and supersymmetry breaking will be introduced, which naturally leads to the definition of the Witten index.

Finally, we apply these techniques to the special case of the supersymmetric harmonic oscillator.

1.3 Statement of Authorship

The theory part of this project follows several texts mentioned in the References section. Several omitted intermediate proofs have been done throughout the whole project (by the author and are attached in the Appendix) to support the theorems. This provides an opportunity to have a glance at the learning process of graduate-level mathematics.

Dr. Johanna Knapp has supervised and pointed out the direction of the whole project. Also, she helped me understand the physics concepts of the content and proofread this report.

2 Introduction

Around the 1970s, people discovered a new type of symmetry in quantum field theory which establishes a relationship between bosons and fermions. We call it supersymmetry. In the first part of the project, we explore a special Lagrangian that is invariant under a supersymmetry variation. Then we analyze several properties like conserved charges and Hamiltonian.

We move from the classical to the quantum system from section 5. We promote the Hamiltonian and the conserved charges to operators on a Hilbert space. We can characterise the Hilbert space in terms of energy levels given by the eigenvalues of the Hamiltonian. From there we explore the additional structure of the supersymmetric Hilbert space and introduce the concept of the Witten index which was first introduced by Edward Witten in [4].

From section 7 we explore how to derive an expression for the ground state of a supersymmetric theory. Also, we apply the theorems to an example called a supersymmetric harmonic oscillator in three ways to show it actually works and the results are consistent.

Notice that we assume $\hbar = 1$ and the mass parameter m to be 1 in all calculations below, therefore they won't show up. Most of the following content follows the book [1]. The required background knowledge about classical mechanics can be found in the book [3].

3 Supersymmetric generalisation of single-variable potential theory

In this section, we will introduce a Lagragian of a theory in one dimention which is invariant under a supersymmetric variation.

3.1 Lagrangian

Consider a Lagrangian given by

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}h'(x)^2 + \frac{i}{2}(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi) - h''(x)\overline{\psi}\psi \tag{1}$$

where x depends on t which is time, and

$$\frac{1}{2}h'(x)^2$$

is the potential $V(x)$ of the system and is a self-interaction term. ψ is superpartner of x and a complex fermion field and $\overline{\psi}$ is its complex conjugate. The third term of the definition is the Lagrangian of the fermion and the last term is the interaction term between the boson x and the fermion which encodes interaction between bosons and fermions. The bosonic part and the fermionic part of the Lagrangian can exist on their own.
Besides, ψ and $\overline{\psi}$ are anti-commuting variables, which means:

$$\psi\overline{\psi} = -\overline{\psi}\psi, \ \ \psi\psi = -\psi\psi, \ \ \overline{\psi\psi} = -\overline{\psi\psi}.$$

As a result:

$$\psi^2 = 0 \tag{2}$$

$$\overline{\psi}^2 = 0. \tag{3}$$

3.2 Supersymmetric variation and invariant action

We will show that the supersymmetric Lagrangian is invariant under a special variation called supersymmetric variation or supersymmetry. Consider an infinitesimal variation of the fields described by

$$\delta x = \epsilon\overline{\psi} - \overline{\epsilon}\psi \tag{4}$$

$$\delta\psi = \epsilon(i\dot{x} + h'(x)) \tag{5}$$

$$\delta\overline{\psi} = \overline{\epsilon}(-i\dot{x} + h'(x)) \tag{6}$$

where ϵ is a complex fermionic parameter. Under this variation, the Lagrangian and action change by (See A.1 for the full proof):

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \delta L dt \\ &= \int_{t_1}^{t_2} \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial \psi}\delta\psi + \frac{\partial L}{\partial \dot{\psi}}\delta\dot{\psi} + \frac{\partial L}{\partial \overline{\psi}}\delta\overline{\psi} + \frac{\partial L}{\partial \dot{\overline{\psi}}}\delta\dot{\overline{\psi}} dt \\ &= i\int_{t_1}^{t_2} \frac{d}{dt}\left(h'(x)(\epsilon\overline{\psi} + \overline{\epsilon}\psi)\right) dt \\ &= 0\end{aligned} \tag{7}$$

This means the action is invariant as long as the boundary variation vanishes:

$$\psi(t_1) = \psi(t_2) = 0 \tag{8}$$

$$\overline{\psi}(t_1) = \overline{\psi}(t_2) = 0 \tag{9}$$

$$x(t_1) = x(t_2) = 0. \tag{10}$$

Hence, the system has a symmetry associated with this variation.

Besides, the square of the fermionic transformation is proportional to the time derivative (See A.3 for the full proof):

$$[\delta_1, \delta_2]x = 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{x} \tag{11}$$

$$[\delta_1, \delta_2]\psi = 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{\psi} \tag{12}$$

We call such a variation supersymmetry.

4 Conserved charges corresponding to the supersymmetry

Now we want to find the conserved charges under supersymmetry. By Noether's theorem such conserved charges must exist, and to find them we use the Noether procedure. The Noether procedure states that if we take the variation parameter to be time-dependent and calculate the variation of the action functional again, then the coefficient of the t-dependent variation parameter is the expression of the conserved charge.

In practice, we replace symmetry parameter ϵ with $\epsilon(t)$ in calculating δS. This gives (See A.4 for the full proof):

$$\delta S = \int_{t_1}^{t_2} (-i\dot{\epsilon}Q - i\dot{\overline{\epsilon}}\overline{Q})dt, \tag{13}$$

where

$$Q = \overline{\psi}(i\dot{x} + h'(x)) \tag{14}$$
$$\overline{Q} = \psi(-i\dot{x} + h'(x)) \tag{15}$$

are the conserved charges associated with supersymmetry, and are called supercharges.

5 Quantisation of the system

In this section, we will move to the quantum system. We promote the variables to operators on a Hilbert space. We will also discuss the relation between Q, $\overline{Q}$ and H as operators.

5.1 Canonical commutation relations

By moving to the quantum system we impose the canonical (anti-)commutation relation given by:

$$[x, p] = i \tag{16}$$
$$\{\psi, \overline{\psi}\} = 1 \tag{17}$$

and all other (anti-)commutators vanishing (= 0). The brackets are defined as:

$$[A, B] = AB - BA \tag{18}$$
$$\{A, B\} = AB + BA. \tag{19}$$

The former is called commutator, the latter is called anti-commutator. In quantising the system we consider x, p, ψ and $\overline{\psi}$ as operators acting on elements of a Hilbert space.

5.2 Conjugate momenta and Hamiltonian

To find the energy levels of the system we need to compute the Hamiltonian. First, we compute the conjugate momenta:

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{20}$$
$$\pi_\psi = \frac{\partial L}{\partial \dot{\psi}} = \frac{i}{2}\overline{\psi} \tag{21}$$
$$\pi_{\overline{\psi}} = \frac{\partial L}{\partial \dot{\overline{\psi}}} = \frac{i}{2}\psi \tag{22}$$

where we can compute the Hamiltonian using its definition (See A.5 for the full proof):

$$\begin{aligned} H &= p\dot{x} + \pi_\psi\dot{\psi} + \pi_{\overline{\psi}}\dot{\overline{\psi}} - L \\ &= \frac{1}{2}p + \frac{1}{2}h'(x)^2 + \frac{1}{2}h''(x)(\overline{\psi}\psi - \psi\overline{\psi}) \end{aligned} \tag{23}$$

5.3 Representation of the operators in Hilbert space

To complete the quantization we must first find a representation of the operators. For the bosonic variable, the Hilbert space is square-normalizable wave functions ($L^2(\mathbb{R}, \mathbb{C})$). We define the operators acting on such a function $\Psi(x)$ to be:

$$\hat{x}\Psi(x) = x\Psi(x) \tag{24}$$

and

$$\hat{p}\Psi(x) = -i\frac{d}{dx}\Psi(x) \tag{25}$$

so that $\hat{x}$ and $\hat{p}$ satisfy (16).

For the fermionic variable, the Hilbert space is two-dimensional. We choose ψ to be the "lowering operator" and define a vacuum state $|0\rangle$ which is annihilated by it:

$$\psi|0\rangle = 0 \tag{26}$$

Then by applying the "raising operator" $\overline{\psi}$ multiple times on vacuum states we can generate a tower of states. Since the "rasing operator" $\overline{\psi}$ satisfies $\overline{\psi}^2 = 0$, then there are only two states:

$$|0\rangle, \overline{\psi}|0\rangle. \tag{27}$$

So the fermionic Hilbert space is a two-dimensional space spanned by these two states as a basis. The operator ψ and $\overline{\psi}$ can be represented by:

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \overline{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{28}$$

so that they satisfy (17). The total Hilbert space is thus given by:

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}) \otimes |0\rangle \oplus L^2(\mathbb{R}, \mathbb{C}) \otimes \overline{\psi}|0\rangle \tag{29}$$

which combines the fermionic and bosonic parts of the physical system

5.4 Commutation relations between H, Q and $\overline{Q}$

From what we have we can show that the conserved charges Q and $\overline{Q}$ commute with the Hamiltonian (See A.6 for the full proof):

$$[H, Q] = [H, \overline{Q}] = 0 \tag{30}$$

Another consequence is that supercharges obey the anti-commutation relation:

$$\{Q, \overline{Q}\} = 2H \tag{31}$$

6 General structure of a supersymmetric Hilbert space and Supersymmetric index

In this section, we further analysis the supersymmetric theory which will lead us to the concept of the Witten index.

6.1 Fermion number operator

The fermion number operator is defined as:

$$F = \overline{\psi}\psi \tag{32}$$

and we define:

$$\mathcal{H}^B = L^2(\mathbb{R}, \mathbb{C})|0\rangle \tag{33}$$

$$\mathcal{H}^F = L^2(\mathbb{R}, \mathbb{C})\overline{\psi}|0\rangle \tag{34}$$

to be the space of bosonic and fermionic states respectively. As a result, the operator F has an eigenvalue 0 on $\mathcal{H}^B$ and an eigenvalue 1 on $\mathcal{H}^F$ which gives a $\mathbb{Z}_2$ grading on $\mathcal{H}$. We introduce an operator $(-1)^F$ which we will use later.

6.2 Decomposition of Hilbert space in terms of energy levels

It is easy to prove that(See A.7 for the full proof):

$$[F, \psi] = -\psi, [F, \overline{\psi}] = \overline{\psi} \tag{35}$$

which gives:

$$[F, Q] = Q, [F, \overline{Q}] = \overline{Q} \tag{36}$$

As a result, the supercharges map one subspace to the other:

$$\begin{aligned} Q, \overline{Q} &: \mathcal{H}^B \to \mathcal{H}^F \\ Q, \overline{Q} &: \mathcal{H}^F \to \mathcal{H}^B \end{aligned} \tag{37}$$

Using the positive-definiteness of the energy, the Hamiltonian H is a non-negative operator (as there is no negative energy):

$$H = \frac{1}{2}\{Q, \overline{Q}\} \geq 0 \tag{38}$$

Also, a state is a zero-energy state under H if and only if it is annihilated by Q and $\overline{Q}$:

$$H|\alpha\rangle = 0 \Leftrightarrow Q|\alpha\rangle = \overline{Q}|\alpha\rangle = 0 \tag{39}$$

Due to the non-negativity of Hamiltonian, a zero energy state is also a ground state of H. We call such states *supersymmetric ground states*. The Hilbert space can be decomposed in terms of eigenspaces of the Hamiltonian with eigenvalues E_n:

$$\mathcal{H} = \bigoplus_{n=0,1,\dots} \mathcal{H}_{(n)}, \quad H|_{\mathcal{H}_{(n)}} = E_n \tag{40}$$

with the convention that $E_0 = 0$. If there is no zero energy state we set $\mathcal{H}_{(0)} = 0$. Since Q and $\overline{Q}$ satisfy (30), they preserve the energy level:

$$Q, \overline{Q} : \mathcal{H}_{(n)} \to \mathcal{H}_{(n)} \tag{41}$$

Since every energy level can be decomposed into bosonic and fermionic subspaces:

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} \oplus \mathcal{H}^F_{(n)} \tag{42}$$

then by (37) we have:

$$Q, \overline{Q} : \mathcal{H}^B_{(n)} \to \mathcal{H}^F_{(n)} \tag{43}$$

$$Q, \overline{Q} : \mathcal{H}^F_{(n)} \to \mathcal{H}^B_{(n)} \tag{44}$$

6.3 Ground states and Witten index

Consider the operator defined by:

$$\mathcal{Q} := Q + \overline{Q} \tag{45}$$

which satisfies:

$$\mathcal{Q}^2 = 2H \tag{46}$$

by (31). Since H is invertible on states with positive energy, $\mathcal{Q}$ is an invertible operator that maps $\mathcal{H}^B_{(n)}$ to $\mathcal{H}^F_{(n)}$ and vise versa for $n \in \mathbb{Z}_{>0}$. This defines an isomorphism:

$$\mathcal{H}^B_{(n)} \cong \mathcal{H}^F_{(n)} \tag{47}$$

for $n \in \mathbb{Z}_{>0}$. This means there is an equal number of bosonic and fermionic states at non-zero energy levels. As for $\mathcal{H}_{(0)}$, $\mathcal{Q}^2 = 2E_0 = 0$. So this does not lead to an isomorphism, and the bosonic and fermionic supersymmetric ground states do not have to be paired.

This means the difference d in the number of total bosonic states and the number of total fermionic states is invariant. We can define d as:

$$d = \mathrm{Tr}(-1)^F e^{-\beta H} \tag{48}$$

where H can be replaced by the eigenvalue of each subspace, β is a parameter, Tr stands for trace over the Hilbert space, and $(-1)^F$ has been introduced in section 6.1. This invariant d is called *Witten index* and is denoted by $\mathrm{Tr}(-1)^F$. Since the positive energy states come in pairs, so they will cancel out, and (48) becomes:

$$\dim\mathcal{H}^B_{(0)} - \dim\mathcal{H}^F_{(0)} = \mathrm{Tr}(-1)^F e^{-\beta H} \tag{49}$$

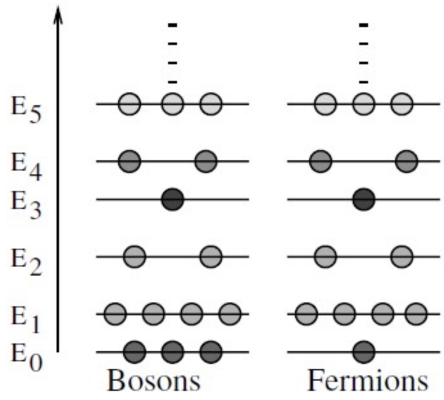

Figure 1: Example of a supersymmetric Hilbert space with Witten index +2

7 Expression of the supersymmetric ground state

Now we compute an expression for the supersymmetric ground state. The supercharges are represented in the $(|0\rangle, \overline{\psi}|0\rangle)$ basis as:

$$Q = \overline{\psi}(ip + h'(x)) = \begin{pmatrix} 0 & 0 \\ d/dx + h'(x) & 0 \end{pmatrix} \tag{50}$$

$$\overline{Q} = \psi(-ip + h'(x)) = \begin{pmatrix} 0 & -d/dx + h'(x) \\ 0 & 0 \end{pmatrix} \tag{51}$$

by (28). To find the gound state, we are looking for an element in $\mathcal{H}$ which is annihilated by the supercharges. Assume $\Psi = f_1(x)|0\rangle + f_2(x)\overline{\psi}|0\rangle$ which is annihilated by Q and $\overline{Q}$:

$$Q\Psi = \begin{pmatrix} 0 & 0 \\ d/dx + h'(x) & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left(\frac{d}{dx} + h'(x)\right) f_1(x) = 0 \tag{52}$$

$$\overline{Q}\Psi = \left(-\frac{d}{dx} + h'(x)\right) f_2(x) = 0. \tag{53}$$

Solving the differential equations gives:

$$f_1(x) = c_1 e^{-h(x)} \tag{54}$$

$$f_2(x) = c_2 e^{h(x)} \tag{55}$$

However, it is not yet clear whether $f_1, f_2 \in L^2(\mathbb{R}, \mathbb{C})$ ($||f_1||_2 < \infty$ and $||f_2||_2 < \infty$). This depends on the behaviors of $h(x)$ at infinity. We consider 3 different asymptotic behaviours of $h(x)$. (We assume polynomial growth of $|h(x)|$ at large x in physics)

7.0.1 Case 1: $h(x) \to -\infty$ as $x \to -\infty$ and $h(x) \to \infty$ as $x \to \infty$

This also applies to the opposite case where the sign of $h(x)$ is flipped. Both $e^{h(x)}$ and $e^{-h(x)}$ are diverging in either one of the infinities. So $||f_1||_2$ and $||f_2||_2$ doesn't exist in $\mathbb{R}$ if $c_1, c_2 \neq 0$ and there is no supersymmetric ground state. As a result, the Witten index is 0:

$$\mathrm{Tr}(-1)^F = 0. \tag{56}$$

7.0.2 Case 2: $h(x) \to \infty$ as $x \to \pm\infty$

$e^{h(x)}$ is diverging in both infinities and $e^{-h(x)}$ decays rapidly at infinity. So f_1 is normalizable and there is one supersymmetric ground state given by:

$$\Psi = c_1 e^{-h(x)} |0\rangle \tag{57}$$

which is in $\mathcal{H}^B$. Thus Witten index is:

$$\mathrm{Tr}(-1)^F = 1. \tag{58}$$

7.0.3 Case 3: $h(x) \to -\infty$ as $x \to \pm\infty$

$e^{-h(x)}$ is diverging in both infinities and $e^{h(x)}$ decays rapidly at infinity. So f_2 is normalizable and there is one supersymmetric ground state given by:

$$\Psi = c_2 e^{h(x)} \overline{\psi} |0\rangle \tag{59}$$

which is in $\mathcal{H}^F$. Thus Witten index is:

$$\mathrm{Tr}(-1)^F = -1. \tag{60}$$

8 Supersymmetric harmonic oscillator

Consider an example called "symmetric harmonic oscillator", where:

$$h(x) = \frac{\omega}{2} x^2 \tag{61}$$

The parameter ω is the frequency of the oscillator and $\omega \in \mathbb{R}$. As a result:

$$V(x) = \frac{1}{2} h'(x)^2 = \frac{\omega^2}{2} x^2 \tag{62}$$

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega [\overline{\psi}, \psi] \tag{63}$$

$$= H_{osc} + H_f, \tag{64}$$

where:

$$H_{osc} = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2 \tag{65}$$

$$H_f = \frac{1}{2} \omega [\overline{\psi}, \psi]. \tag{66}$$

We are now using 3 different methods to find the Witten index of the system.

8.0.1 Method 1

First, we will compute the supersymmetric ground state. When $\omega > 0$, $h(x)$ satisfies case 2 in section 7.0.2. So the supersymmetric ground state is given by:

$$\Psi_{\omega>0} = c_1 e^{-\frac{1}{2}\omega x^2}|0\rangle \tag{67}$$

and the Witten index is +1. As for $\omega < 0$, $h(x)$ satisfies case 3 (7.0.3) so the supersymmetric ground states is:

$$\Psi_{\omega<0} = c_2 e^{-\frac{1}{2}|\omega|x^2}\overline{\psi}|0\rangle \tag{68}$$

and the Witten index is −1.

8.0.2 Method 2

Secondly, we approach this conclusion by computing all the energy states of the Hamiltonian. As we can see that H_{osc} in (65) is the same as the Hamiltonian for the simple harmonic oscillator and has eigenvalues [2]:

$$\frac{|\omega|}{2}, \frac{|\omega|}{2} + |\omega|, \frac{|\omega|}{2} + 2|\omega|, ... \tag{69}$$

For the fermionic part of the Hamiltonian, H_f can be represented as the matrix:

$$\frac{\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{70}$$

with respect to the $(|0\rangle, \overline{\psi}|0\rangle)$ basis. This has eigenvalues $\pm\frac{|\omega|}{2}$.

So the eigenvalues of total Hamiltonian with $\omega > 0$ are:

$$\begin{aligned} &\text{bosonic states:} \quad 0, |\omega|, 2|\omega|, ... \\ &\text{fermionic states:} \quad |\omega|, 2|\omega|, ... \end{aligned}$$

which indicates a zero energy bosonic ground state while all positive energy states have boson-fermion pairing. This indicates a +1 Witten index as shown before.

As for $\omega < 0$:

$$\begin{aligned} &\text{bosonic states:} \quad |\omega|, 2|\omega|, ... \\ &\text{fermionic states:} \quad 0, |\omega|, 2|\omega|, ... \end{aligned}$$

which indicates a zero energy fermionic ground state and a −1 Witten index.

8.0.3 Method 3

Thirdly, we can directly calculate the Witten index $\text{Tr}(-1)^F$. Recall that the Hilbert space factorizes as:

$$\begin{aligned}\mathcal{H} &= (L^2(\mathbb{R},\mathbb{C}) \otimes |0\rangle) \oplus (L^2(\mathbb{R},\mathbb{C}) \otimes \overline{\psi}|0\rangle) \\ &= L^2 \otimes \mathbb{C}^2 \end{aligned} \tag{71}$$

where:

$$\begin{aligned} L^2 &= L^2(\mathbb{R},\mathbb{C}) \\ \mathbb{C}^2 &= \mathbb{C}|0\rangle \oplus \mathbb{C}\overline{\psi}|0\rangle \end{aligned} \tag{72}$$

Since H_{osc} acts on L^2 non-trivially and H_f acts on $\mathbb{C}^2$ non-trivially, the Witten index is defined by:

$$\begin{aligned} \text{Tr}(-1)^F &:= \text{Tr}_{\mathcal{H}}[(-1)^F e^{-\beta H}] \qquad (73) \\ &= \text{Tr}_{L^2} e^{-\beta H_{osc}} \times \text{Tr}_{\mathbb{C}^2}[(-1)^F e^{-\beta H_f}]. \end{aligned}$$

Calculating each part using the eigenvalues computed in method 2:

$$\text{Tr}_{L^2} e^{-\beta H_{osc}} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})|\omega|} \tag{74}$$

$$= e^{-\frac{\beta|\omega|}{2}} \frac{1}{1-e^{-\beta|\omega|}} \tag{75}$$

$$= \frac{1}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} \tag{76}$$

$$\text{Tr}_{\mathbb{C}^2}[(-1)^F e^{-\beta H_f}] = e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}. \tag{77}$$

Thus

$$\text{Tr}(-1)^F = \frac{e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \frac{\omega}{|\omega|} = \pm 1 \tag{78}$$

as expected. Hence all three methods give the same Witten index. Another important result is that we have shown in method 3 that the result of the Witten index is independent of β and that the calculation will generate an integer value in the end.

9 Conclusion

We have developed a supersymmetric generalization to an example of a field theory in one dimension. We have shown how introducing supersymmetry leads to extra structure on the Hilbert space of the system. From there we built up an understanding of the Witten index and computed it using a simple example called a supersymmetric harmonic oscillator.

We can further explore the mathematical notion of a Morse function that comes up in this context, and the concepts can also be applied to further, more complicated models such as the supersymmetric version of a multi-particle system on theories in higher dimensions.

A Some proofs

In this appendix, we give detailed computations to derive the results shown in the main text. This is not given in [1].

A.1 Supersymmetric variation of the action

We now want to prove the (7) that the action is invariant under the supersymmetric variation (4). Firstly we have

$$\delta\dot{x} = \epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi} \tag{79}$$

$$\delta\dot{\psi} = \epsilon(i\ddot{x} + h''(x)\dot{x}) \tag{80}$$

$$\delta\dot{\overline{\psi}} = \overline{\epsilon}(-i\ddot{x} + h''(x)\dot{x}) \tag{81}$$

by

$$\delta\dot{x} = \frac{d}{dt}\delta x \tag{82}$$

$$\delta\dot{\psi} = \frac{d}{dt}\delta\psi \tag{83}$$

$$\delta\dot{\overline{\psi}} = \frac{d}{dt}\delta\overline{\psi} \tag{84}$$

from (4). By definition of Lagrangian in (1):

$$\frac{\partial L}{\partial x} = -h'(x)h''(x) - h'''(x)\overline{\psi}\psi \tag{85}$$

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{86}$$

$$\frac{\partial L}{\partial \psi} = -\frac{i}{2}\dot{\overline{\psi}} - h''(x)\overline{\psi} \tag{87}$$

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{i}{2}\overline{\psi} \tag{88}$$

$$\frac{\partial L}{\partial \overline{\psi}} = -\frac{i}{2}\dot{\psi} + h''(x)\psi \tag{89}$$

$$\frac{\partial L}{\partial \dot{\overline{\psi}}} = \frac{i}{2}\psi \tag{90}$$

Using boundary conditions (8) we have:

$$
\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L dt \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \frac{\partial L}{\partial \psi}\delta \psi + \frac{\partial L}{\partial \dot{\psi}}\delta \dot{\psi} + \frac{\partial L}{\partial \overline{\psi}}\delta \overline{\psi} + \frac{\partial L}{\partial \dot{\overline{\psi}}}\delta \dot{\overline{\psi}} dt \\
&= \int_{t_1}^{t_2} -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
&\qquad + \left(-\frac{i}{2}\dot{\overline{\psi}} - h''(x)\overline{\psi}\right)\delta\psi + \left(\frac{i}{2}\overline{\psi}\right)\delta\dot{\psi} \\
&\qquad + \left(-\frac{i}{2}\dot{\psi} + h''(x)\psi\right)\delta\overline{\psi} + \left(\frac{i}{2}\psi\right)\delta\dot{\overline{\psi}} dt \\
&= \int_{t_1}^{t_2} -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
&\qquad + \left(\frac{i}{2}\dot{\overline{\psi}} - i\dot{\overline{\psi}} - h''(x)\overline{\psi}\right)\delta\psi + \left(\frac{i}{2}\overline{\psi}\right)\delta\dot{\psi} \\
&\qquad + \left(\frac{i}{2}\dot{\psi} - i\dot{\psi} + h''(x)\psi\right)\delta\overline{\psi} + \left(\frac{i}{2}\psi\right)\delta\dot{\overline{\psi}} dt \\
&= \int_{t_1}^{t_2} -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
&\qquad + \left(-i\dot{\overline{\psi}} - h''(x)\overline{\psi}\right)\delta\psi + \left(-i\dot{\psi} + h''(x)\psi\right)\delta\overline{\psi} dt \\
&\quad + \int_{t_1}^{t_2} \left(\frac{i}{2}\dot{\overline{\psi}}\right)\delta\psi + \left(\frac{i}{2}\overline{\psi}\right)\delta\dot{\psi} dt \qquad\qquad (= 0) \\
&\quad + \int_{t_1}^{t_2} \left(\frac{i}{2}\dot{\psi}\right)\delta\overline{\psi} + \left(\frac{i}{2}\psi\right)\delta\dot{\overline{\psi}} dt \qquad\qquad (= 0) \\
&= \int_{t_1}^{t_2} -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
&\qquad + \left(-i\dot{\overline{\psi}} - h''(x)\overline{\psi}\right)\epsilon(i\dot{x} + h'(x)) \\
&\qquad + \left(-i\dot{\psi} + h''(x)\psi\right)\overline{\epsilon}(-i\dot{x} + h'(x))dt \\
&= \int_{t_1}^{t_2} -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
&\qquad + \left(\dot{\overline{\psi}}\epsilon\dot{x} - h'(x)h''(x)\overline{\psi}\epsilon - ih'(x)\dot{\overline{\psi}}\epsilon - ih''(x)\dot{x}\overline{\psi}\epsilon\right) \\
&\qquad + \left(-\dot{\psi}\overline{\epsilon}\dot{x} + h'(x)h''(x)\psi\overline{\epsilon} - ih'(x)\dot{\psi}\overline{\epsilon} - ih''(x)\dot{x}\psi\overline{\epsilon}\right) dt
\end{aligned}
$$

Re-arranging terms gives:

$$\begin{aligned}
\delta S = \int_{t_1}^{t_2} & -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
& + \dot{x}(-\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) \\
& - h'(x)h''(x)(-\epsilon\overline{\psi} + \overline{\epsilon}\psi) \\
& + ih'(x)(\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) + ih''(x)\dot{x}(\epsilon\overline{\psi} + \overline{\epsilon}\psi)dt \\
= i \int_{t_1}^{t_2} & h'(x)(\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) + h''(x)\dot{x}(\epsilon\overline{\psi} + \overline{\epsilon}\psi)dt \\
= i \int_{t_1}^{t_2} & \frac{d}{dt}\left(h'(x)(\epsilon\overline{\psi} + \overline{\epsilon}\psi)\right)dt \\
= 0 &
\end{aligned} \tag{91}$$

This holds since in physics, one defines:

$$\left[\epsilon\overline{\psi} + \overline{\epsilon}\psi\right]_{t_1}^{t_2} = 0 \tag{92}$$

A.2 Equation of motion

Here we want to compute the equation of motion. Firstly note that an arbitrary variation of action is given by:

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L dt \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial \psi}\delta\psi + \frac{\partial L}{\partial \dot{\psi}}\delta\dot{\psi} + \frac{\partial L}{\partial \overline{\psi}}\delta\overline{\psi} + \frac{\partial L}{\partial \dot{\overline{\psi}}}\delta\dot{\overline{\psi}}dt
\end{aligned}$$

Using that

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}}\delta\dot{x} &= \left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x \\
&= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x
\end{aligned}$$

Thus

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \frac{\partial L}{\partial x}\delta x - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x + \frac{\partial L}{\partial \psi}\delta\psi - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right)\delta\psi + \frac{\partial L}{\partial \overline{\psi}}\delta\overline{\psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\overline{\psi}}}\right)\delta\overline{\psi}dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x + \left(\frac{\partial L}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right)\right)\delta\psi + \left(\frac{\partial L}{\partial \overline{\psi}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\overline{\psi}}}\right)\right)\delta\overline{\psi}dt
\end{aligned}$$

Since we want to have the equation of motion, we set the variation of action to 0:

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x + \left(\frac{\partial L}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right)\right)\delta\psi + \left(\frac{\partial L}{\partial \overline{\psi}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\overline{\psi}}}\right)\right)\delta\overline{\psi}dt \\
&= 0.
\end{aligned}$$

So

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \tag{93}$$

$$\frac{\partial L}{\partial \psi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) \tag{94}$$

$$\frac{\partial L}{\partial \overline{\psi}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\overline{\psi}}}\right). \tag{95}$$

This gives:

$$-h'(x)h''(x) - h'''(x)\overline{\psi}\psi = \ddot{x} \tag{96}$$

$$-i\dot{\overline{\psi}} - h''(x)\overline{\psi} = 0 \tag{97}$$

$$-i\dot{\psi} + h''(x)\psi = 0 \tag{98}$$

by inserting everything in (85). So by the least action principle, we have the equation of motion of the system.

A.3 Fermionic transformation

A.3.1 Square of fermionic transformatioin

We want to show (11) holds here. We have

$$\begin{aligned}
[\delta_1, \delta_2]x &= \delta_1\delta_2 x - \delta_2\delta_1 x \\
&= \delta_1(\epsilon_2\overline{\psi} - \overline{\epsilon_2}\psi) - \delta_2(\epsilon_1\overline{\psi} - \overline{\epsilon_1}\psi) \\
&= \epsilon_2\overline{\epsilon_1}(-i\dot{x} + h'(x)) - \overline{\epsilon_2}\epsilon_1(i\dot{x} + h'(x)) \\
&\quad + \overline{\epsilon_1}\epsilon_2(i\dot{x} + h'(x)) - \epsilon_1\overline{\epsilon_2}(-i\dot{x} + h'(x)) \\
&= \epsilon_2\overline{\epsilon_1}(-i\dot{x} + h'(x)) + \epsilon_1\overline{\epsilon_2}(i\dot{x} + h'(x)) \\
&\quad - \epsilon_2\overline{\epsilon_1}(i\dot{x} + h'(x)) - \epsilon_1\overline{\epsilon_2}(-i\dot{x} + h'(x)) \\
&= -2\epsilon_2\overline{\epsilon_1}i\dot{x} + 2\epsilon_1\overline{\epsilon_2}i\dot{x} \\
&= 2i(\epsilon_1\overline{\epsilon_2} - \epsilon_2\overline{\epsilon_1})\dot{x}
\end{aligned}$$

and

$$\begin{aligned}
[\delta_1, \delta_2]\psi &= \delta_1\delta_2\psi - \delta_2\delta_1\psi \\
&= \delta_1\epsilon_2(i\dot{x} + h'(x)) - \delta_2\epsilon_1(i\dot{x} + h'(x)) \\
&= \epsilon_2\delta_1(i\dot{x} + h'(x)) - \epsilon_1\delta_2(i\dot{x} + h'(x)) \\
&= i\epsilon_2\delta_1\dot{x} + \epsilon_2\delta_1 h'(x) - i\epsilon_1\delta_2\dot{x} - \epsilon_1\delta_2 h'(x) \\
&= i\epsilon_2\delta_1\dot{x} + \epsilon_2 h''(x)\delta_1 x - i\epsilon_1\delta_2\dot{x} - \epsilon_1 h''(x)\delta_2 x \\
&= i\epsilon_2\delta_1\dot{x} + \epsilon_2 h''(x)\delta_1 x - i\epsilon_1\delta_2\dot{x} - \epsilon_1 h''(x)\delta_2 x \\
&= i\epsilon_2(\epsilon_1\dot{\overline{\psi}} - \overline{\epsilon}_1\dot{\psi}) + \epsilon_2 h''(x)(\epsilon_1\overline{\psi} - \overline{\epsilon}_1\psi) \\
&\quad - i\epsilon_1(\epsilon_2\dot{\overline{\psi}} - \overline{\epsilon}_2\dot{\psi}) - \epsilon_1 h''(x)(\epsilon_2\overline{\psi} - \overline{\epsilon}_2\psi)
\end{aligned}$$

Using (96):

$$h''(x) = -i\frac{\dot{\overline{\psi}}}{\overline{\psi}} = i\frac{\dot{\psi}}{\psi} \tag{99}$$

gives:

$$\begin{aligned}
[\delta_1, \delta_2]\psi &= i\epsilon_2(\epsilon_1\dot{\overline{\psi}} - \overline{\epsilon}_1\dot{\psi}) + \epsilon_2 h''(x)(\epsilon_1\overline{\psi} - \overline{\epsilon}_1\psi) \\
&\quad - i\epsilon_1(\epsilon_2\dot{\overline{\psi}} - \overline{\epsilon}_2\dot{\psi}) - \epsilon_1 h''(x)(\epsilon_2\overline{\psi} - \overline{\epsilon}_2\psi) \\
&= i\epsilon_2(\epsilon_1\dot{\overline{\psi}} - \overline{\epsilon}_1\dot{\psi}) + \epsilon_2(-i\epsilon_1\dot{\overline{\psi}} - \overline{\epsilon}_1 i\dot{\psi}) \\
&\quad - i\epsilon_1(\epsilon_2\dot{\overline{\psi}} - \overline{\epsilon}_2\dot{\psi}) - \epsilon_1(-i\epsilon_2\dot{\overline{\psi}} - i\overline{\epsilon}_2\dot{\psi}) \\
&= -2i\epsilon_2\overline{\epsilon}_1\dot{\psi} + 2i\epsilon_1\overline{\epsilon}_2\dot{\psi} \\
&= 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{\psi}.
\end{aligned} \tag{100}$$

A.4 Conserved charges

Now we derive (14) here. To compute the conserved charges using the Noether procedure we first take:

$$\epsilon = \epsilon(t).$$

Then

$$\begin{aligned}
\delta S = \int_{t_1}^{t_2} & -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
& + \dot{x}(\dot{\epsilon}\overline{\psi} - \dot{\overline{\epsilon}}\psi) \\
& + \left(-\frac{i}{2}\dot{\overline{\psi}} - h''(x)\overline{\psi}\right)\delta\psi + \left(\frac{i}{2}\overline{\psi}\right)\delta\dot{\psi} \\
& + \left(-\frac{i}{2}\dot{\psi} + h''(x)\psi\right)\delta\overline{\psi} + \left(\frac{i}{2}\psi\right)\delta\dot{\overline{\psi}}dt \\
= \int_{t_1}^{t_2} & -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
& + \dot{x}(\dot{\epsilon}\overline{\psi} - \dot{\overline{\epsilon}}\psi) \\
& + \left(-i\dot{\overline{\psi}} - h''(x)\overline{\psi}\right)\epsilon(i\dot{x} + h'(x)) \\
& + \left(-i\dot{\psi} + h''(x)\psi\right)\overline{\epsilon}(-i\dot{x} + h'(x))dt \\
= \int_{t_1}^{t_2} & -h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + \dot{x}(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}) \\
& + \dot{x}(-\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) \\
& - h'(x)h''(x)(-\epsilon\overline{\psi} + \overline{\epsilon}\psi) \\
& + \dot{x}(\dot{\epsilon}\overline{\psi} - \dot{\overline{\epsilon}}\psi) \\
& + ih'(x)(\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) + ih''(x)\dot{x}(\epsilon\overline{\psi} + \overline{\epsilon}\psi)dt \\
= \int_{t_1}^{t_2} & \dot{x}(\dot{\epsilon}\overline{\psi} - \dot{\overline{\epsilon}}\psi) + ih'(x)(\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) + ih''(x)\dot{x}(\epsilon\overline{\psi} + \overline{\epsilon}\psi)dt
\end{aligned} \tag{101}$$

Using Assumption (92) gives

$$\begin{aligned}\int_{t_1}^{t_2} & h'(x)(\epsilon\dot{\overline{\psi}} + \overline{\epsilon}\dot{\psi}) + h''(x)\dot{x}(\epsilon\overline{\psi} + \overline{\epsilon}\psi)dt \\ &= [h'(x)(\epsilon\overline{\psi} + \overline{\epsilon}\psi)]_{t1}^{t2} - \int_{t_1}^{t_2} h'(x)(\dot{\epsilon}\overline{\psi} + \dot{\overline{\epsilon}}\psi)dt \\ &= \int_{t_1}^{t_2} -h'(x)(\dot{\epsilon}\overline{\psi} + \dot{\overline{\epsilon}}\psi)dt\end{aligned}$$

So (101) becomes:

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \dot{x}(\dot{\epsilon}\overline{\psi} - \dot{\overline{\epsilon}}\psi) - ih'(x)(\dot{\epsilon}\overline{\psi} + \dot{\overline{\epsilon}}\psi)dt \\ &= \int_{t_1}^{t_2} (-i)i(\dot{\epsilon}\overline{\psi} - \dot{\overline{\epsilon}}\psi)\dot{x} + (-i)h'(x)(\dot{\epsilon}\overline{\psi} + \dot{\overline{\epsilon}}\psi)dt \\ &= \int_{t_1}^{t_2} (-i)\dot{\epsilon}\overline{\psi}(i\dot{x} + h'(x)) + (-i)\dot{\overline{\epsilon}}\psi(-i\dot{x} + h'(x)) \\ &= \int_{t_1}^{t_2} (-i\dot{\epsilon}Q - i\dot{\overline{\epsilon}}\overline{Q})dt \end{aligned} \tag{102}$$

where

$$Q = \overline{\psi}(i\dot{(x)} + h'(x)) \tag{103}$$

$$\overline{Q} = \psi(-i\dot{(x)} + h'(x)) \tag{104}$$

are the expressions of conserved charges.

A.5 Hamiltonian

We now want to compute the Hamiltonian (23) using (20):

$$\begin{aligned}H &= p\dot{x} + \pi_\psi\dot{\psi} + \pi_{\overline{\psi}}\dot{\overline{\psi}} - L \\ &= p^2 + \frac{i}{2}(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi) - L \\ &= \frac{1}{2}p + \frac{1}{2}h'(x)^2 + h''(x)\overline{\psi}\psi \\ &= \frac{1}{2}p + \frac{1}{2}h'(x)^2 + \frac{1}{2}h''(x)\overline{\psi}\psi - \frac{1}{2}h''(x)\psi\overline{\psi} \\ &= \frac{1}{2}p + \frac{1}{2}h'(x)^2 + \frac{1}{2}h''(x)(\overline{\psi}\psi - \psi\overline{\psi}).\end{aligned} \tag{105}$$

A.6 (Anti-)commutation relation between H, Q, $\overline{Q}$

A.6.1 $\mathbb{Z}_2$-graded bracket

First, we have a $\mathbb{Z}_2$ grading $|\ \ |$ defined as:

$$|x| = \begin{cases} 0 & \text{if } x \text{ is commuting} \\ 1 & \text{if } x \text{ is anti-commuting} \end{cases} \tag{106}$$

The (anti)commuting brackets are defined as:

$$[A, B]_{\pm} = AB - (-1)^{|A||B|}BA \tag{107}$$

For convenience we denote $[\ ,\ ]_{\pm}$ as $[\ ,\ ]$ in the following calculation.

A.6.2 Composite commuting brackets

One can easily show that:

$$\begin{aligned}
[AB, C] &= A[B, C] + (-1)^{|B||C|}[A, C]B \\
[A, BC] &= [A, B]C + (-1)^{|A||B|}B[A, C] \\
[AB, CD] &= A[B, C]D + s_1 AC[B, D] + s_2[A, C]DB + s_3 C[A, D]B
\end{aligned}$$

where:

$$\begin{aligned}
s_1 &= (-1)^{|B||D|} \\
s_2 &= (-1)^{|B||C|+|B||D|} \\
s_3 &= (-1)^{|B||C|+|B||D|+|A||C|}.
\end{aligned}$$

A.6.3 $[H, Q]$ and $[H, \overline{Q}]$

We now want to prove (30). Using (16) we have:

$$[H, Q] = \frac{i}{2}[p^2, \overline{\psi}p] + \frac{1}{2}[p^2, \overline{\psi}h'(x)] + \frac{i}{2}[h'(x)^2, \overline{\psi}p] + \frac{1}{2}[h'(x)^2, \overline{\psi}h'(x)] + i[h''(x)\overline{\psi}\psi, \overline{\psi}p] + [h''(x)\overline{\psi}\psi, \overline{\psi}h'(x)]$$

$$\begin{aligned}
\frac{i}{2}[p^2, \overline{\psi}p] &= 0 \\
\frac{1}{2}[p^2, \overline{\psi}h'(x)] &= -ih''(x)\overline{\psi}p \\
\frac{i}{2}[h'(x)^2, \overline{\psi}p] &= -h''(x)\overline{\psi}h'(x) \\
\frac{1}{2}[h'(x)^2, \overline{\psi}h'(x)] &= 0 \\
i[h''(x)\overline{\psi}\psi, \overline{\psi}p] &= ih''(x)\overline{\psi}p \\
[h''(x)\overline{\psi}\psi, \overline{\psi}h'(x)] &= h''(x)\overline{\psi}h'(x).
\end{aligned}$$

By using (35) and:

$$[p, h'(x)] = -ih''(x) \tag{108}$$

this means:

$$[H, Q] = -ih''(x)\overline{\psi}p + ih''(x)\overline{\psi}p - h''(x)\overline{\psi}h'(x) + h''(x)\overline{\psi}h'(x) \tag{109}$$

$$= 0 \tag{110}$$

Following the same steps will give the result:

$$[H, \overline{Q}] = 0. \tag{111}$$

Also:

$$[Q, \overline{Q}] = 2H. \tag{112}$$

A.7 Fermion number operator

In this section, we will prove the commutation relation between the fermion number operator and ψ and $\overline{\psi}$ (in (35)).

Using (16) we have:

$$\begin{aligned}
[F, \psi] &= [\overline{\psi}\psi, \psi] \\
&= \overline{\psi}\psi\psi - \psi\overline{\psi}\psi \\
&= \overline{\psi}\psi\psi + \overline{\psi}\psi\psi - \overline{\psi}\psi\psi - \psi\overline{\psi}\psi \\
&= \overline{\psi}\{\psi, \psi\} - \{\psi, \overline{\psi}\}\psi \\
&= -\psi
\end{aligned} \tag{113}$$

and

$$\begin{aligned}
[F, \overline{\psi}] &= [\overline{\psi}\psi, \overline{\psi}] \\
&= \overline{\psi}\psi\overline{\psi} - \overline{\psi}\overline{\psi}\psi \\
&= \overline{\psi}\psi\overline{\psi} + \overline{\psi}\overline{\psi}\psi - \overline{\psi}\overline{\psi}\psi - \overline{\psi}\overline{\psi}\psi \\
&= \overline{\psi}\{\psi, \overline{\psi}\} - \{\overline{\psi}, \overline{\psi}\}\psi \\
&= \overline{\psi}.
\end{aligned} \tag{114}$$

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